

ADVANCED MATHS DICTIONARY

WITH WORKED ANSWERS

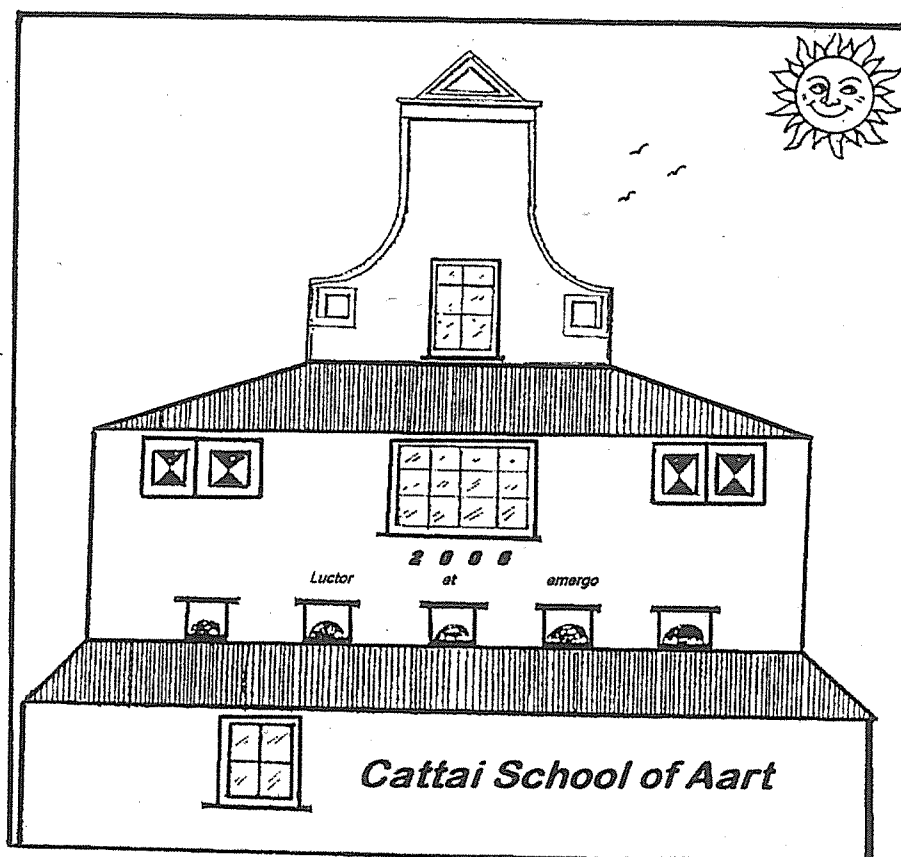
BACK TO BASICS

An Educational Revolution

Aart Bark

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Note: Index numbers refer to Computer Reading

H.S.C. LATERAL REVISION

Go Four Times Across the Topics

Plain Maths

Educational and Practical

Plain means that the simplest, accepted symbols have been used.

Plain means that there is no sleep-inducing text.

Plain means no unnecessary frills. If people use Tim for Timothy, they should feel comfortable with Pythagoras instead of the clumsy "Theorem of Pythagoras" or worse, "Pythagoras's Theorem" (I can't even say it properly). Likewise you'll find isosceles, equilateral, opp. instead of vertically opposite.

Plain means that both Differentiation and Integration have successfully been explained with only a few words and diagrams. The routine preceeds the First Principles so as not to empty out the baby with the bath water.

Plain means that there are no pictures of famous mathematicians or technical equipment; this is a mathsbook, not a museum.

Plain means that there are no cartoons, clowns or silly professors with blackboards, desperately trying to convince students that Maths can be fun. This book has nothing to do with Australia's Wonderland.

Educational

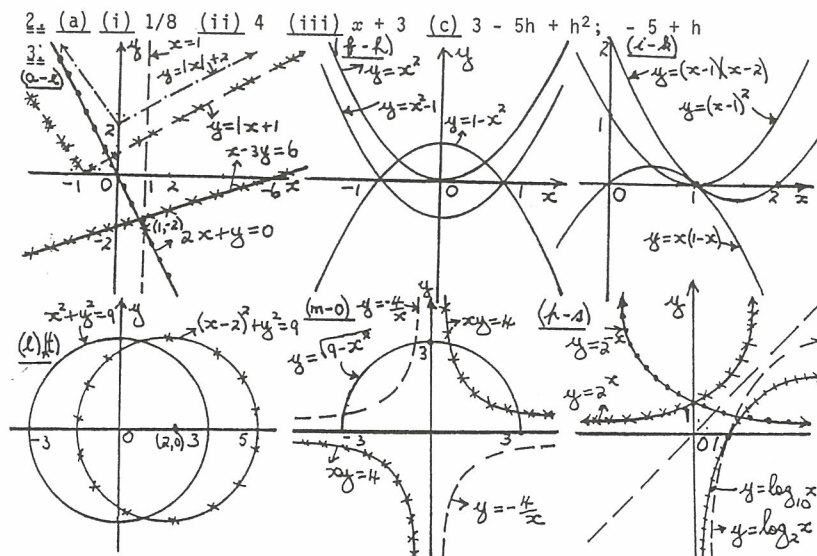
1. Data is given in a shorthand fashion, much like a Memo.
2. In most cases, there are No Instructions; it teaches initiative, attention and concentration, the prerequisites in Memory Training. Knowing what to do is part of the lesson; a bushfire doesn't come with instructions how to extinguish it.
3. Algebraic developments in answers have been reduced to a minimum so as not to obscure the main plot; it teaches the student to be inquisitive and to practise the details. Understanding is only the beginning, successfully doing, the target. Looking at a sum is not good enough; you don't become a comedian by watching the Goonshow.
4. Although the victims of a nuclear explosion won't need to calculate the probability of survival, they still need to put two and two together. That's why Major Institutions still use a Standard Maths test to screen applicants, because they don't want people who say, "I don't know, I only work here." With the **right attitude**, Maths is eminently suitable to help develop

All the necessary skills to successfully tackle any job or activity.

It is an abstract and exact subject based on facts discovered by Humans, free from their personal emotions, opinions, likes, dislikes and beliefs. It is incorruptible and dependable; what was true three thousand years ago is still true today. New discoveries are way beyond the K-12 curriculum.

Practical

- Practical** means that there is absolutely no difference between professional methods and those shown to the student.
- Practical** means that he doesn't have to be recycled once he leaves school.
- Practical** means a quick transition from understanding to routine.
- Practical** means the usage of Professional Memory Training techniques: Visualisation, Verbal Rehearsal and Association. Necessary formulas are "filled in mentally."
- Practical** means fostering the awareness of patterns: "All the psychological and physiological evidence that we have suggests that, with regard to perception, the human mind is a pattern-making and pattern-using system." (de Bono)
- Practical** Differentiation and Integration means that all mesmerising and pedantic concentrations of symbolic jargon have been abolished and replaced by simple recipes enabling the student to write down answers immediately.
- Practical** Trigonometry means that the variable is left out in the calculation when there is only one; efficiency and clarity.
- Practical** means that there is no need to waste time by filling blackboards and workbooks.
- Practical** means that the complete course is contained in only 250 pages; 860 exercises rather than 5000 or more. They can easily be worked through in the allocated 240 hours.
- Practical** means feeding the given information directly into the calculator, but not like a zombie, of course. Pressing 3x4 in the middle of a complex calculus problem is like doing up shoelaces during an Olympic race. A truck driver doesn't write down that the light is red; he puts his foot on the brake.
- Practical** means that all exercises come with worked solutions. There is no forest of answers at the back, so you won't get dizzy.



Edecaytional

In the past, audacious pioneers who challenged existing practices or beliefs were either beheaded, burnt to death, imprisoned or exiled for life. I would certainly have suffered a similar ordeal. Fortunately, the Board of Studies recognises that the aims and objectives of the syllabus may be achieved in a variety of ways and by the application of many different techniques. Success in the achievement of these aims and objectives is the concern of the Board which does not, however, either stipulate or evaluate specific teaching methods.

(Board of Senior School Studies: Mathematics Syllabus).

Unfortunately, this very lifeline is cut by an extremely Ambiguous and Retarding instruction given to H.S.C. candidates.

All necessary working should be shown.

Necessary to whom?

The question has never been asked and consequently never been answered. There is no definition either.

One teacher came close by saying to his students that they simply had to treat the markers as educated idiots. It's like having to write down that it is three o'clock because the small hand is on the three and the big one on the twelve.

It results in Compulsory incompetence, creates Mentally handicapped students and is therefore in direct conflict with the obviously theoretical aims of encouraging the gifted and the not so gifted.

Adults will invariably reminisce by confirming the fact rather than give it some serious thought as I to say, "Do unto others what was done unto us."

"Answers only" tests as well as the controversial "Multiple Choice" ones disprove that necessity.

<u>"Working or Else"</u> is blackmail:	scribbles and wrong answer, half mark.
No	scribbles and right answer, half mark.

It is used, and therefore abused, to stop students from copying.

It is used to show students where they went wrong; it's like belting a dog for taking off with your socks the week before:

It is used to write the H.S.C. Post mortem because statistics are more important than people.

Testing instead of Teaching. However, merely discussing symptoms is like playing Trivial Pursuit, Panadol is no replacement for plastic surgery.

It is used to market books with deceiving titles such as "Modern Maths" or "New Maths" which only refer to low I.Q. methods, to endless repetitions of topics and to a 400% increase in size to keep students occupied according to Parkinson's Law: work expands to fill the time available for its completion.

Progress is Man's ability to complicate simplicity. Besides, this progress is negatively geared in order to keep up with the decline in students' performance.

The more complex the system, the more open it is to total breakdown.

Standard Exam

H.S.C. exams should be abolished immediately.

Apart from being extremely costly, they are unreliable because of the surrealistic marks the calculation of which would even have baffled Einstein.

They have no educational value whatsoever. They encourage mediocrity.

They are an obsolete remnant of the past that could be compared with the various, pain-inflicting initiation rituals of primitive tribes.

They have created a new breed of pseudo experts and Anti-Stress Course charlatans. How on earth can you be under stress with only five subjects to study for? I did fifteen during World War 11; towards the end, sitting on a pushbike in the living room, book on the handle bars, pedalling to generate light! Worry is like a rocking chair; keeps you going, but gets you nowhere.

This book contains all the material for a Standard Exam.

Knowing questions beforehand is, of course, quite successfully used to obtain a driver's licence.

Topic tests could be given and marked at school, with only **Full Marks** accepted as a pass.

A certificate of competence could then be earned in stages.

It would foster a new type of incentive.

Somebody called the School System a cul-de-sac down which ideas are lured and then quietly strangled.

In that case, the converted teacher and his conscientious disciples should still prefer using this **Manual** because it is the only one that shows the light at the end of the tunnel and not one of an oncoming train either.

Textbook without Text

"Images are better than words, showing better than telling, too much instruction worse than none; conscious trying often produces negative results."

(The inner game of tennis: Gallwey)

How to use it

(With effortless effort:Zen)

The following procedure stops you from collapsing as soon as you have to solve a problem; it is the run before the jump so to speak.

1. **Look at the question.**
2. **Identify the topic** (eye, thought, voice). Saying "Algebra" is like saying "Book" to the librarian. The brain must know before it can act.
3. **Learn to remember the required strategy or recipe.**
4. **Practise, revise, practise again.**
5. Research has shown that only 15 percent of success is due to technical training; the other 85 percent to personality and attitude.

ABSOLUTE VALUES

1.

$$|-3| - |-8|$$

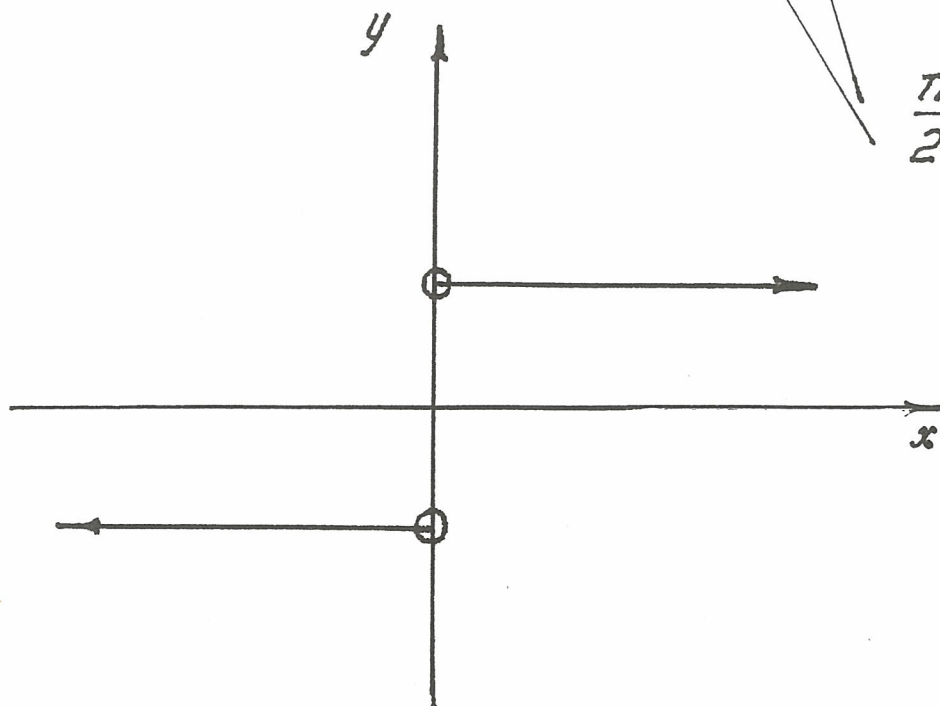
SAY: (MODULUS-3) NEG. (MOD.-8)

ABSOLUTE VALUES ARE ALWAYS

IGNORE —

$$3 - 8 = -5$$

GRAPH $y = \frac{|x|}{x}$



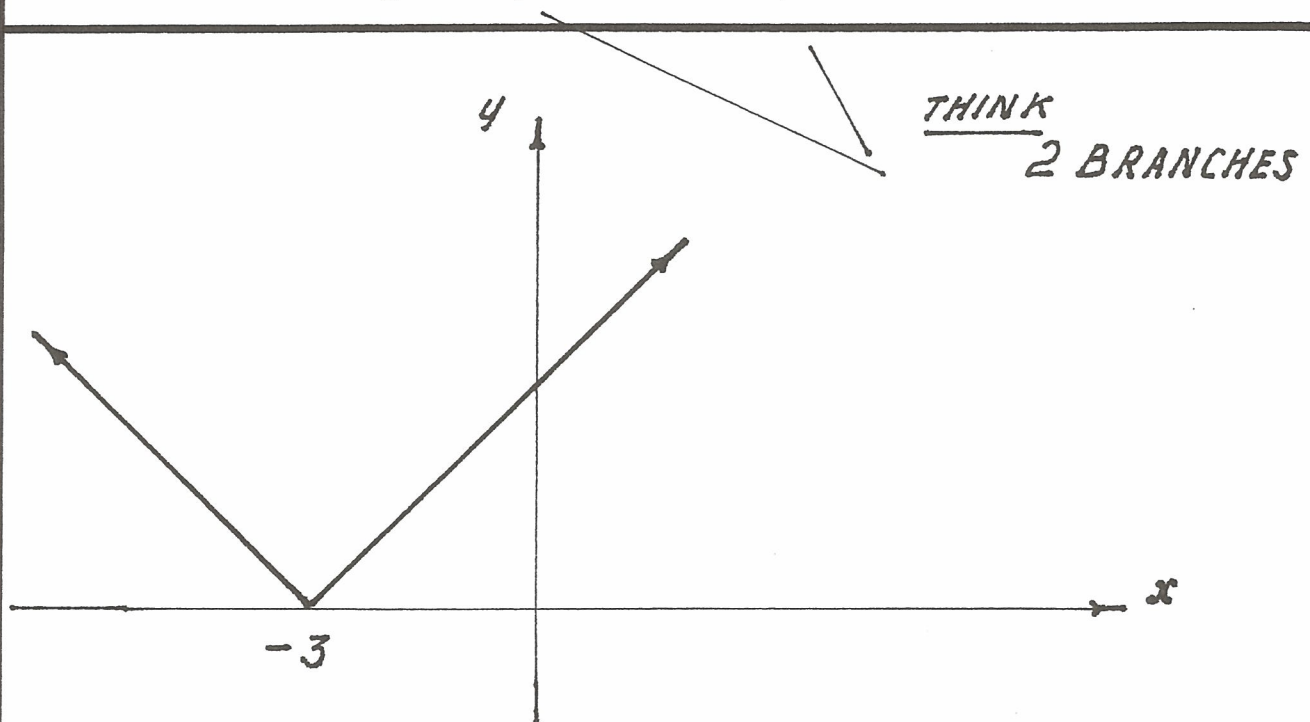
THINK
2 BRANCHES
 $x \neq 0$

ABSOLUTE VALUES

2.

GRAPH

$$y = |x + 3|$$



GRAPH

$$|x| < 4 \text{ AND } x^2 < 16$$

MEANS

x IS LESS THAN 4
FROM
ZERO

ROUTINE PREVIEW
JOINT GRAPH
BETWEEN -4 & 4 (NOT INCL)

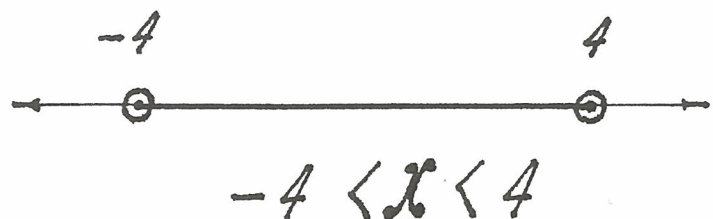
REVISION

$$|x| = 4$$

THINK 2 VALUES
 $x = \pm 4$

SAME GRAPH FOR BOTH

TEST $x = 0$



ABSOLUTE VALUES

3.

$$|x| \gg 3$$

$$x^2 \gg 9$$

MEANS

x IS 3 OR
MORE THAN 3
FROM
ZERO

THINK 2 ANSWERS
POS & NEG 3

ROUTINE PREVIEW
DISJOINT GRAPH

SAME GRAPH FOR BOTH



TEST $x=0$: FALSE

$$|2x+4| < 8$$

$$|2x+4| \gg 8$$

AS YOU SEE IT

CHANGE
SIGN & SYMBOL

AS YOU SEE IT

CHANGE
SIGN & SYMBOL

$$2x+4 < 8$$

$$2x+4 > -8$$

$$2x+4 > 8$$

$$2x+4 \leq -8$$

$$x < 2$$

$$x > -6$$

$$x \gg 2$$

$$x \leq -6$$

WORK OUT SEPARATELY!

WORK OUT SEPARATELY



$$-6 < x < 2$$

READ: x BETWEEN -6 & 2



$$x \leq -6$$

$$x \gg 2$$

$$\{x: x \leq -6 \cup x \gg 2\}$$

ABSOLUTE VALUES

4.

$$4 \leq |2x+4| \leq 8$$

INTERSECTION OF

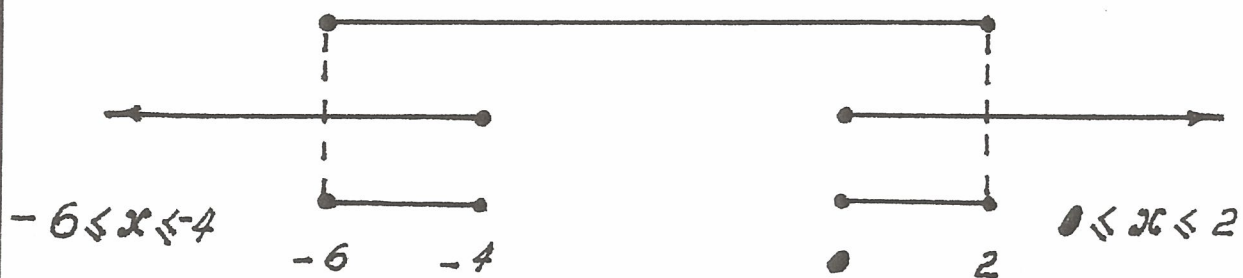
$$|2x+4| \geq 4 \quad \text{AND} \quad |2x+4| \leq 8$$

$$2x+4 \geq 4 \\ x \geq 0$$

$$2x+4 \leq -4 \\ x \leq -4$$

$$2x+4 \leq 8 \\ x \leq 2$$

$$2x+4 \geq -8 \\ x \geq -6$$



$$|x-1| = \underline{2x-5}$$

THE STRATEGY
MENTAL PREPARATION

BEWARE!

MUST BE ≥ 0
CHECK ANSWERS
TO VERIFY

AS YOU SEE IT

CHANGE SIGN

THE

DETAILS PAST ROUTINES

$$x-1 = 2x-5 \\ x=4, \text{ SUB}(2x-5) \checkmark$$

$$x-1 = -2x+5 \\ x=2, \text{ SUB}(2x-5). \text{ N.A.}$$

LEARN THE STRATEGY

PRACTISE THE DETAILS

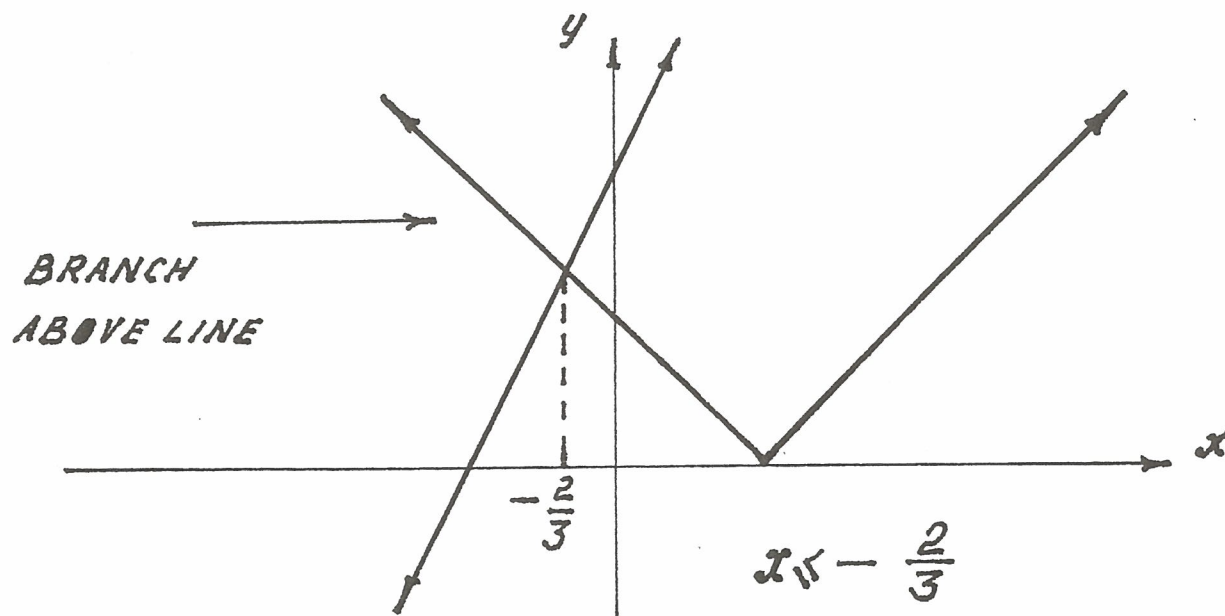
ABSOLUTE VALUES

5.

LOOK
& SEE

$$|x-2| \gg 2x+4$$

REMEMBER: GRAPHIC SOLUTION



LOOK
& SEE

$$|x+2| = |2x-5|$$

REMEMBER:

4 POSSIBILITIES REDUCIBLE TO 2

$$x+2 = 2x-5$$

$$x = 7$$

CHANGE SIGN

$$x+2 = -2x+5$$

$$x = 1$$

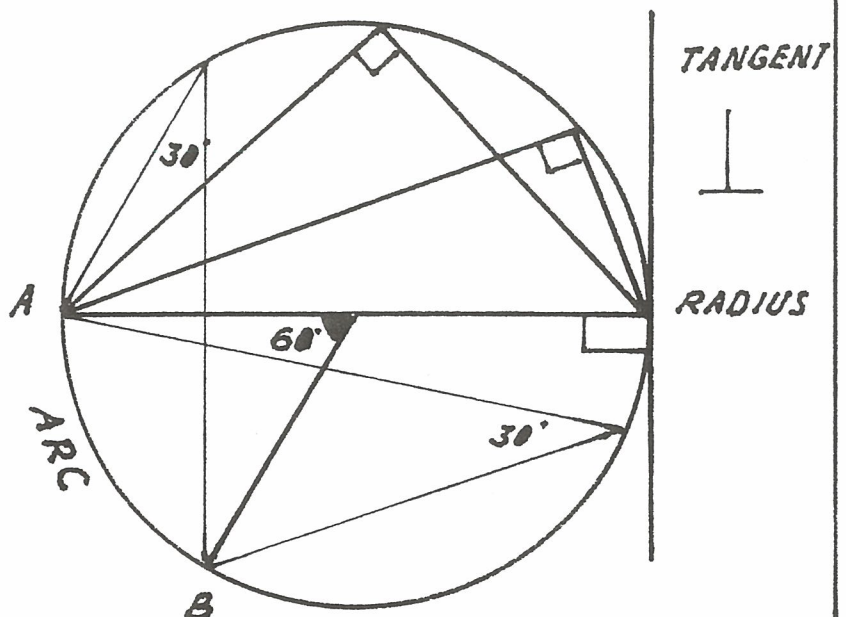
LEARN THE STRATEGY PRACTISE THE DETAILS

6.

LINES & PARTS



ANGLE = ARC



CIRCLES

7

CIRCUMFERENCE AROUND FERRY

IT WAS FOUND THAT C
IS ALWAYS A BIT MORE THAN 3
TIMES THE DIAMETER.
 $D \times \pi$

$$\pi = P \div D = P \text{ FOR PERIMETER}$$

$$\pi = \frac{22}{7} \div 3.14$$

HOWEVER, USE $C = 2R\pi$
BECAUSE OF WORK WITH RADII

AREA

A CIRCLE CAN BE
THOUGHT OF AS AN
INFINITE NUMBER OF
TRIANGLES,
TOTAL BASE $2R\pi$ AND
HEIGHT R

TOTAL AREA

$$R^2 \pi$$

RADII 3, 4, 5, 6, 7, 8 cm

MENTAL ROUTINE

EXPRESS
ANSWERS IN π

FOR C ,

DOUBLE THE R

$6\pi, 8\pi, 10\pi,$

$12\pi, 14\pi, 16\pi$ cm

PUT IN CALCULATOR IF NECESSARY

MENTAL ROUTINE

EXPRESS
ANSWERS IN π

FOR A ,

SQUARE THE R

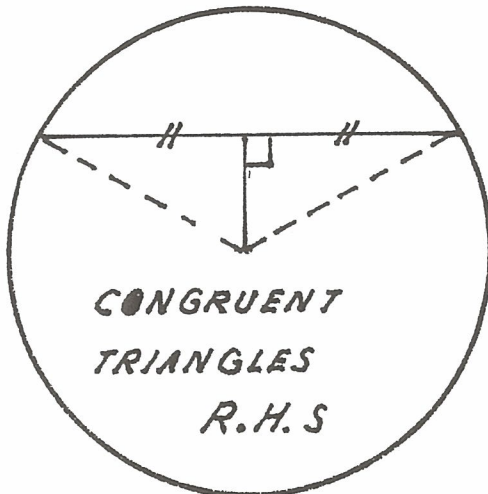
$9\pi, 16\pi, 25\pi,$

$36\pi, 49\pi, 64\pi$ cm²

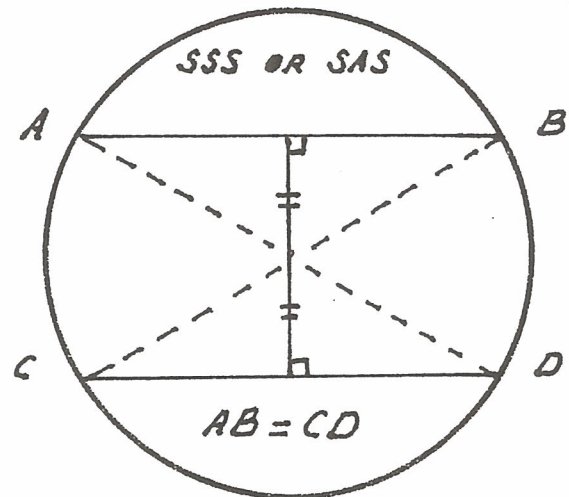
CIRCLES

8.

PROPERTIES



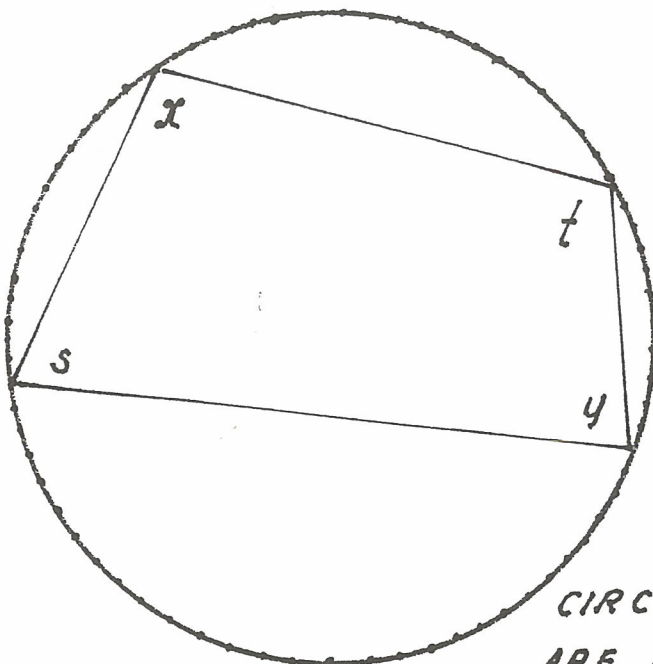
THE PERPENDICULAR
BISECTS THE CHORD
IN 2 CUT & V.V.



EQUAL CHORDS ARE
EQUIDISTANT FROM
THE CENTRE & V.V.

CYCLIC QUADRILATERAL

FOUR SIDES



$$x + y = 180^\circ$$

$$s + t = 180^\circ$$

REMEMBER:

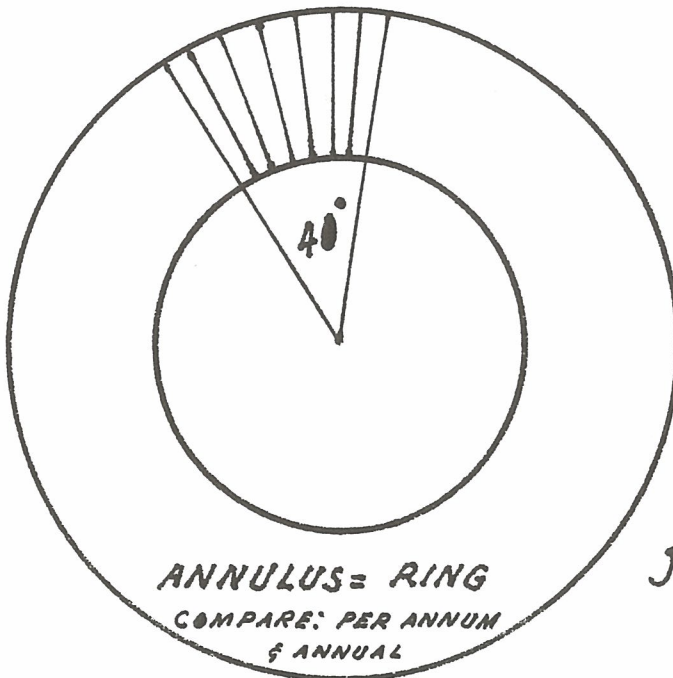
CIRCUMFERENCE ANGLES
ARE $\frac{1}{2}$ THE ARC THEY STAND ON

CIRCLES

9.

CONCENTRIC
WITH (SAME) CENTRE

RADII 3 & 6 cm



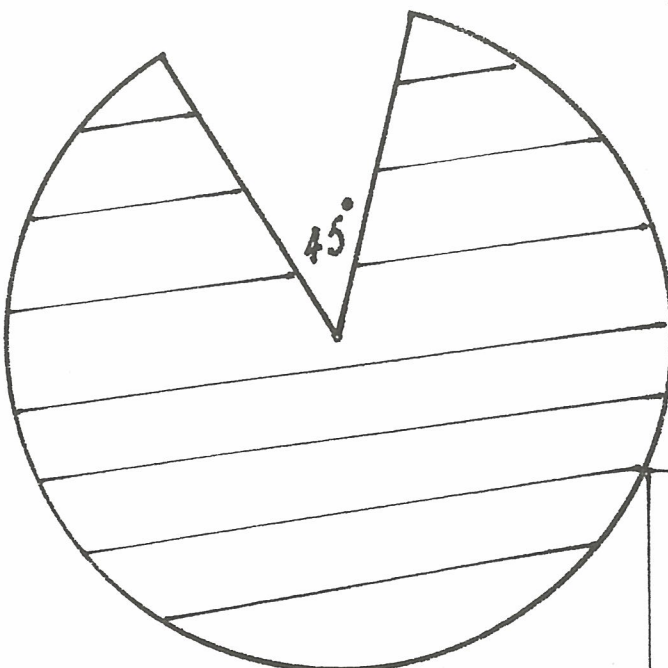
AREA ANNULUS $27\pi \text{ cm}^2$

SHADED AREA $3\pi \text{ cm}^2$

PERIMETER

$$\pi\left(\frac{12}{9} + \frac{6}{9}\right) + 6 = 2\pi + 6 \text{ cm}$$

AREA & PERIMETER SHADED AREA



$R = 8 \text{ cm}$

PERIMETER $\frac{7}{8}$ CIRCLE
+ $2R$
 $(14\pi + 16) \text{ cm}$

AREA $56\pi \text{ cm}^2$

45°

$\frac{1}{8}$

40°

$\frac{1}{9}$

30°

$\frac{1}{12}$

120°

$\frac{1}{3}$

CIRCLES

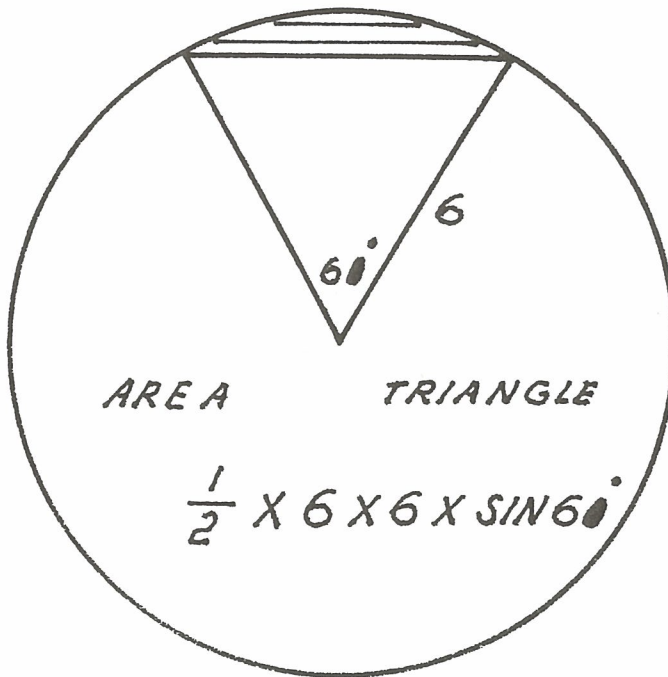
10.

$$R = 6 \text{ cm}$$

PHI

$$\phi = 60^\circ$$

AREA SEGMENT



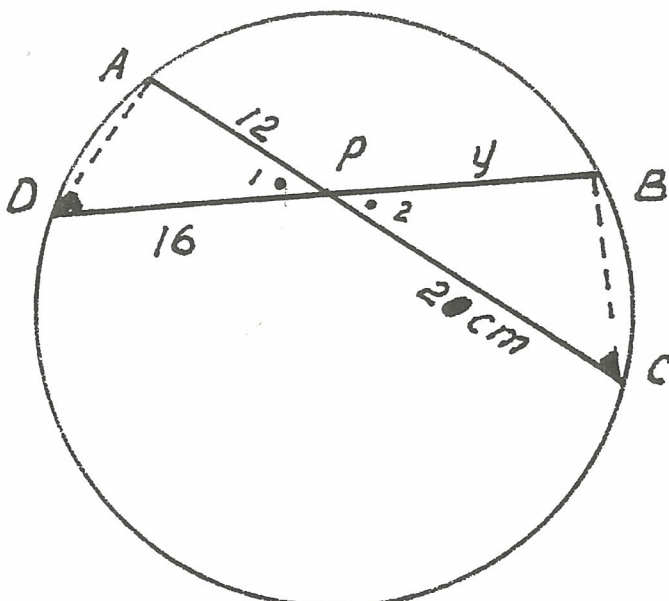
$$\text{AREA SECTOR} = 6\pi$$

$$\text{AREA TRIANGLE} = 9\sqrt{3}$$

$$\text{AREA SEGMENT} \doteq 3.3 \text{ cm}^2$$

INTERSECTING CHORDS

THINK RATIO . USE SIMILAR TRIANGLES
 DRAW — — —



$$P_1 = P_2 \text{ (OPP)}$$

$$\hat{D} = \hat{C} \text{ (ON ARC AB)}$$

$$\triangle ADP \parallel \triangle BCP$$

LETTERS MUST CORRESPOND

$$\frac{16}{20} = \frac{12}{4} = 15 \text{ cm}$$

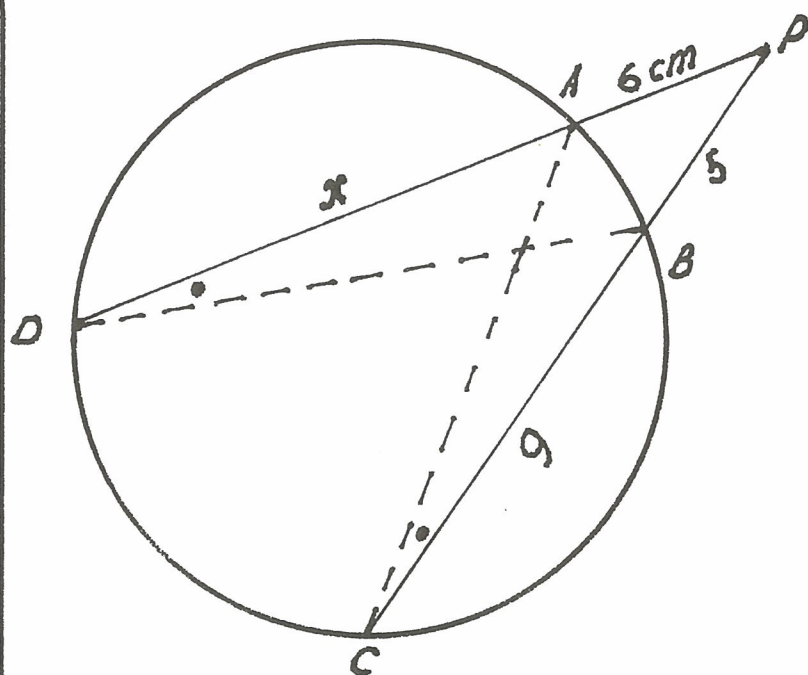
TIMES DIVIDE

CROSS MULTIPLY
 SHORT CUT

CIRCLES

11.

SECANTS FROM EXTERNAL P



THINK **RATIO** &
SIMILAR TRIANGLES

DRAW - - - -

$\hat{P} = \hat{P}$, $\hat{D} = \hat{C}$ (ON ARC AB)
 $\therefore \hat{A} = \hat{B}$
 $\triangle PDB \parallel \triangle PCA$

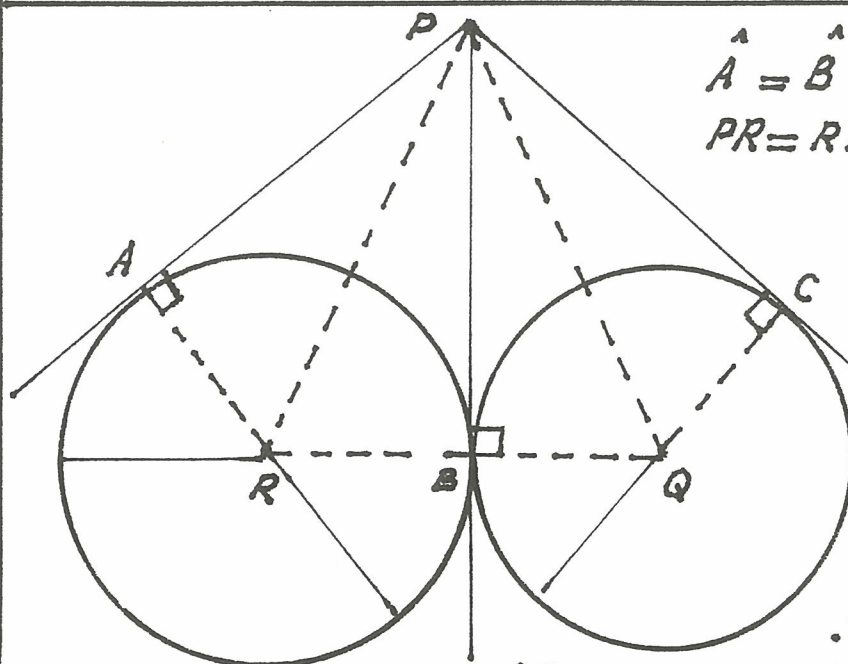
$$\frac{5}{6} = \frac{x+6}{14}$$

CROSS

MULTIPLY $x = 5\frac{2}{3}$ cm

EQUAL TANGENTS

THINK CONGRUENT TRIANGLES. DRAW - - - -



$\hat{A} = \hat{B}$ ($\frac{1}{2}$ SEMICIRCLE) = 90°

$PR = RP$, $AR = BR$ (RADII)

$\therefore \triangle PAR \equiv \triangle PBR$

R.H.S

$\therefore PA = PB$

SIMILARLY $PB = PC$

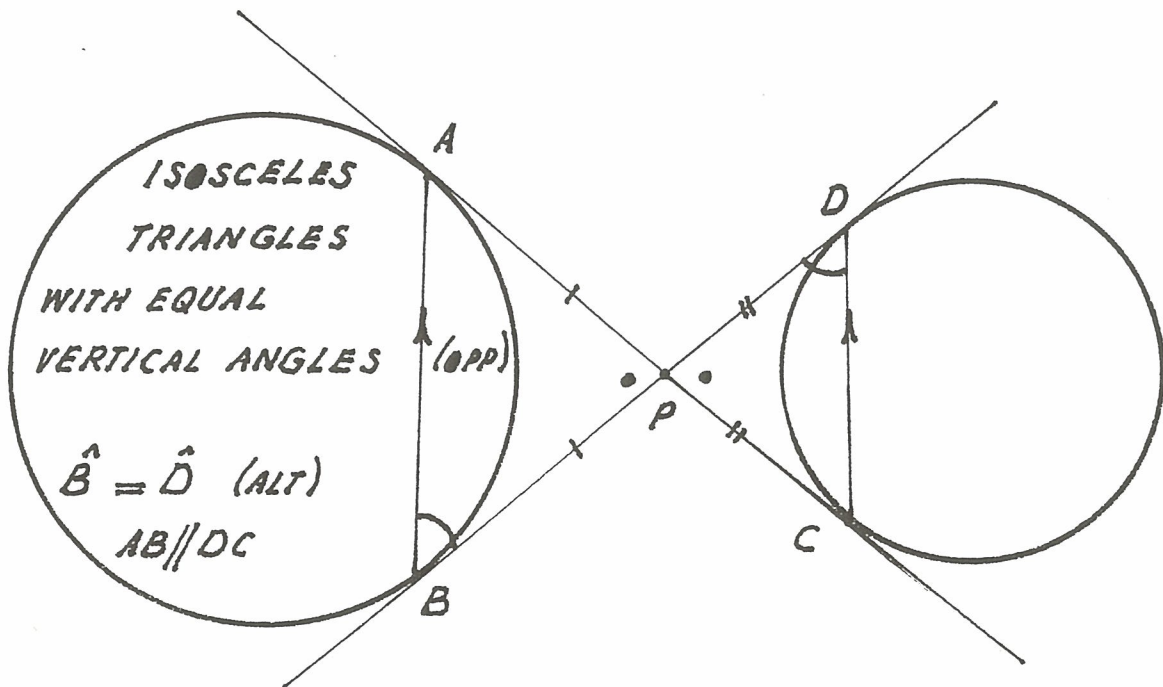
$\therefore PA = PB = PC$ q.e.d.

QUOD ERAT DEMONSTRANDUM

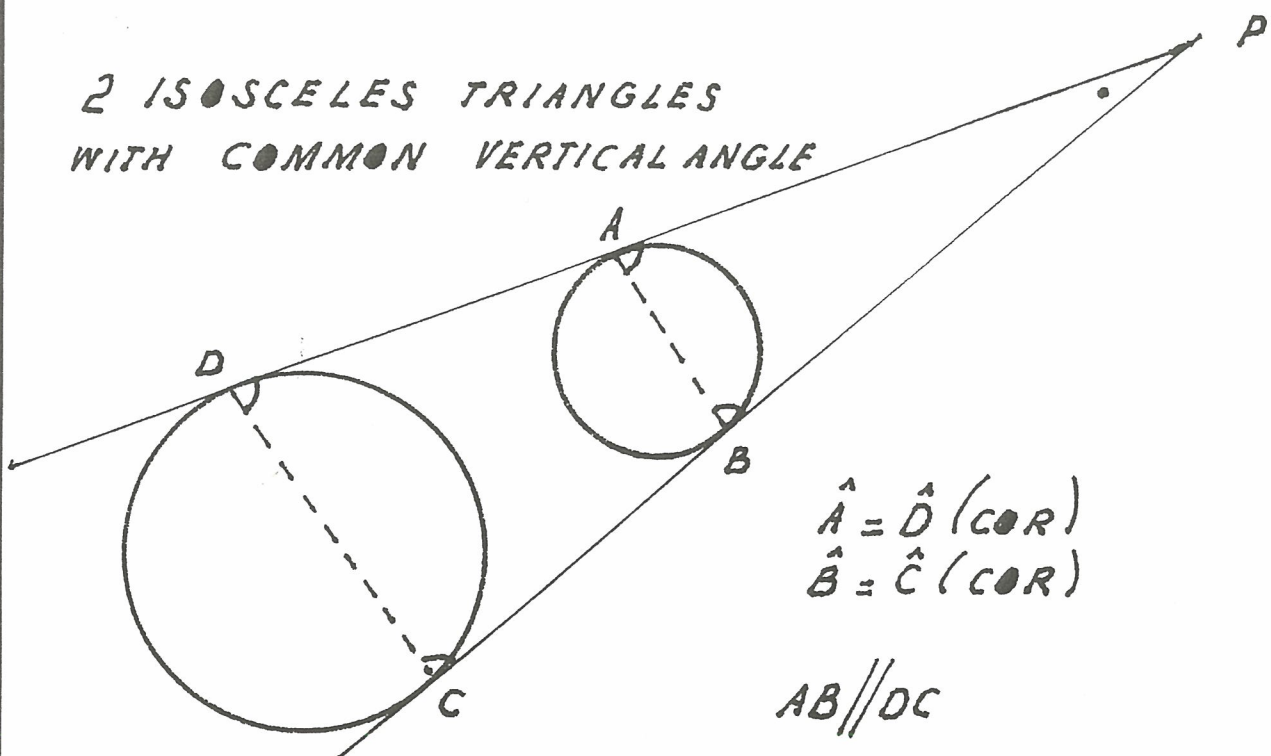
CIRCLES

12.

EQUAL TANGENTS, PARALLEL CHORDS I



EQUAL TANGENTS, PARALLEL CHORDS II

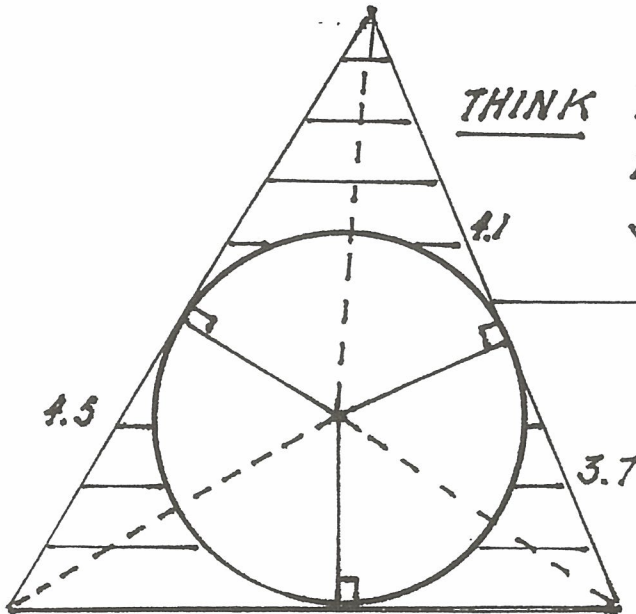


CIRCLES

13.

SHADED AREA

$R = 2.5 \text{ cm}$



THINK EQUAL TANGENTS

TANGENTS $\perp$ RADII

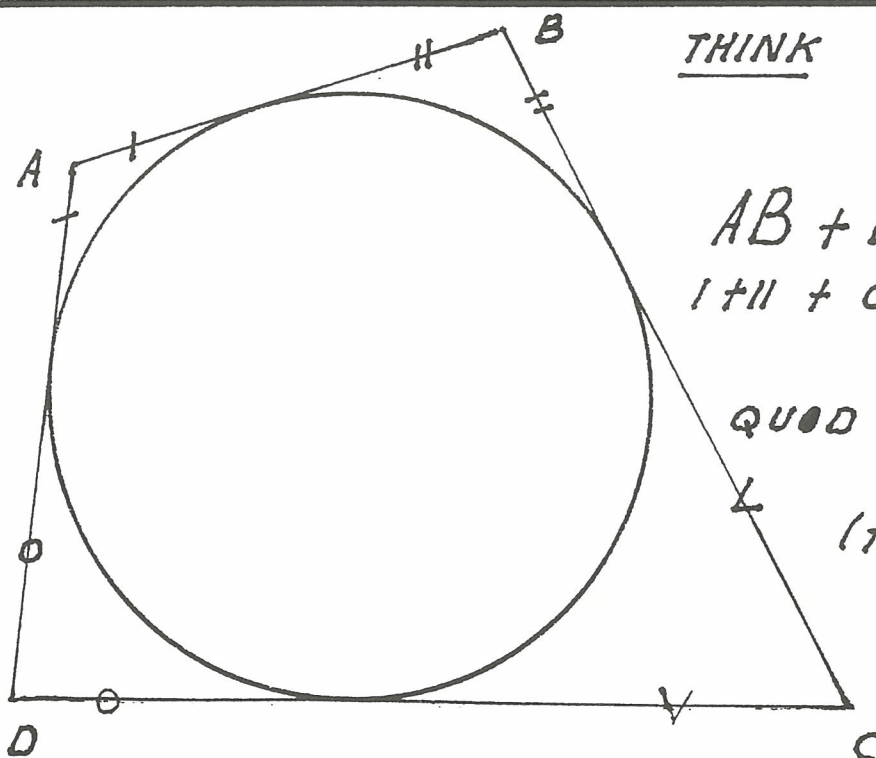
SHADED AREA = TRIANGLE - CIRCLE

$$\frac{\text{AREA TRIANGLE}}{1.25(8.6 + 7.8 + 8.2)}$$

$$\text{AREA CIRCLE} = 6.25\pi$$

$$\text{SHADED AREA} \doteq 11.1 \text{ cm}^2$$

SUMS OPPOSITE SIDES ARE EQUAL



THINK EQUAL TANGENTS

$$AB + DC = AD + BC$$

$$I + H + O + V = I + O + H + V$$

q.e.d.

QUOD ERAT

DEMONSTRANDUM

(THAT'S WHAT WE HAD
TO PROVE)

CIRCLES

14.

EQUATION-CENTRE-RADIUS RELATION

DISCOVER THE SYSTEM

$$(0, 0), R = 6$$

$$x^2 + y^2 = 36$$

$$(5, -3), R = 4$$

$$(x-5)^2 + (y+3)^2 = 16$$

$$(x-3)^2 + y^2 = 100$$

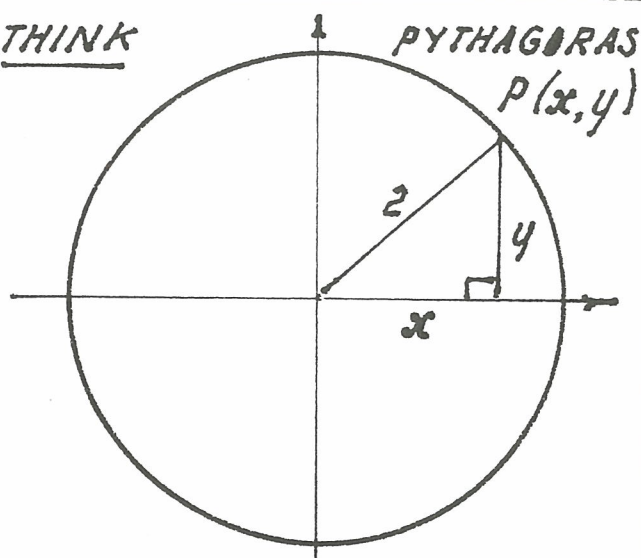
SEE HOW IT WORKS

$$(3, 0), R = 10$$

DIAGRAM

$$x^2 + y^2 = 4$$

THINK



CIRCLES ARE RELATIONS:

FOR EACH x -VALUE,
THERE ARE 2 y -VALUES.

(VERTICAL LINE TEST)

CENTRE $(0,0)$, $R = 2$

CIRCLES

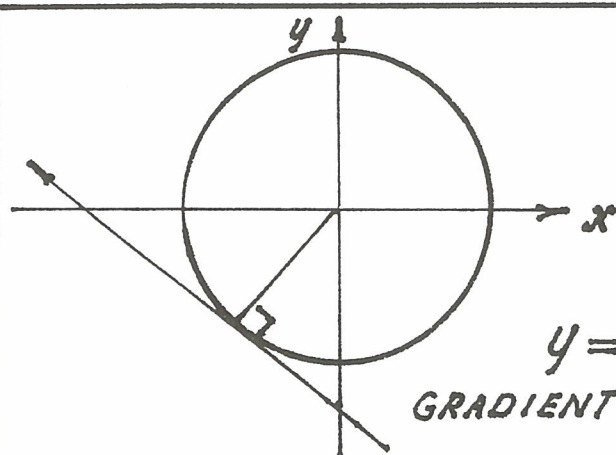
15.

PROVE

$3x + 4y + 10 = 0$ IS TANGENT TO CIRCLE $(0,0)$, $R=2$

THINK DISTANCE CENTRE $\rightarrow$ LINE
 SUB $(0,0)$ IN GENERAL EQ. (IGNORE 0)

$$\frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$$



$$D = \frac{10}{5} = 2 \text{ UNITS } \text{q.e.d.}$$

$$y = -\frac{3}{4}x - \frac{5}{2}$$

GRADIENT-INTERCEPT FORM

$$x^2 + y^2 - 8y + 10x - 8 = 0$$

THINK: CIRCLE
 COMPLETING SQUARES
 HALVE B AND SQUARE TO GET C

COMPENSATE!

$$(x^2 + 10x + 25) - 25 + (y^2 - 8y + 16) - 16 - 8 = 0$$

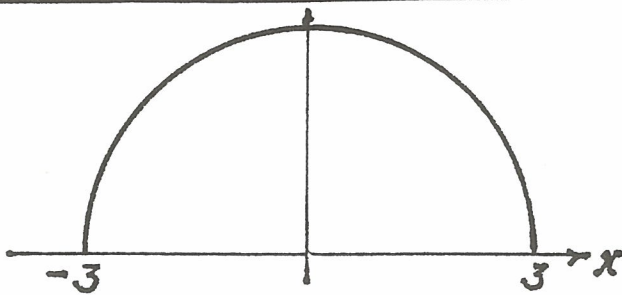
$$(x+5)^2 + (y-4)^2 = 49$$

CENTRE $(-5, 4)$, $R=7$

CIRCLES

16.

SEMI-CIRCLES ARE FUNCTIONS



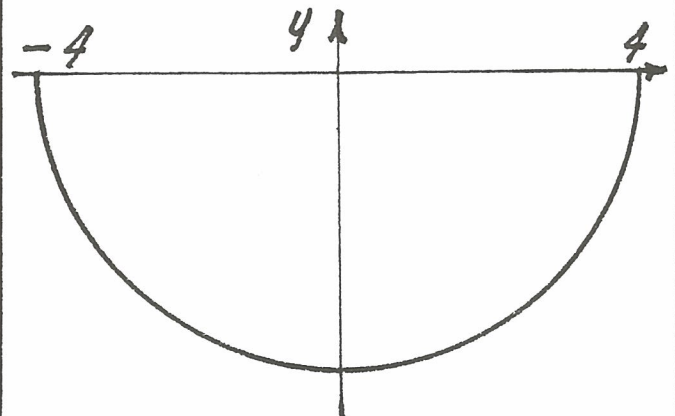
$$y = \sqrt{9 - x^2}$$

UPPER BRANCH

OF $x^2 + y^2 = 9$

$$y = -\sqrt{16 - x^2}$$

LOWER BRANCH
OF $x^2 + y^2 = 16$



IF THE SUM OF 2 RADII IS CONSTANT, THE AREA-SUM IS LEAST WHEN $R_1 = R_2$

THINK DIFFERENTIATION: $A' = 0$

TRANSLATE

$$R_1 + R_2 = C$$

DIFFERENTIATE
WITH RESPECT TO ONE VARIABLE

$$R_2 = C - R_1$$

$$A = \pi(2R_1^2 - 2CR_1 + C^2)$$

$$A' = 2\pi(2R_1 - C) = 0$$

AREA SUM IS LEAST WHEN $R_1 = \frac{1}{2}C$

WHEN $R_1 = R_2$ q.e.d.

CIRCLES

17.

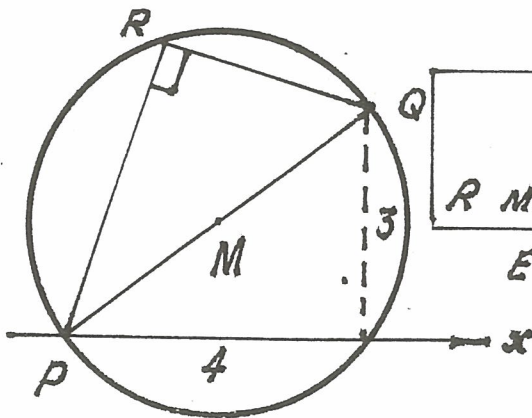
$P(-1, 0), Q(3, 3), R(x, y). PR \perp QR$
 PROVE: LOCUS OF R SATISFIES $x^2 + y^2 - 2x - 3y - 3 = 0$

LOCUS = PATH, LOCATION OF R

↓ STANDARD PROCEDURE:

1. SELECT ONE POSITION FOR THE POINT IN QUESTION
2. WRITE AN EQUATION BY USING AVAILABLE DATA.

↓ THINK CIRCLE



$$PQ = 5, M(1, 1.5)$$

R MUST LIE ON A CIRCLE $R = 2.5$, CENTRE M

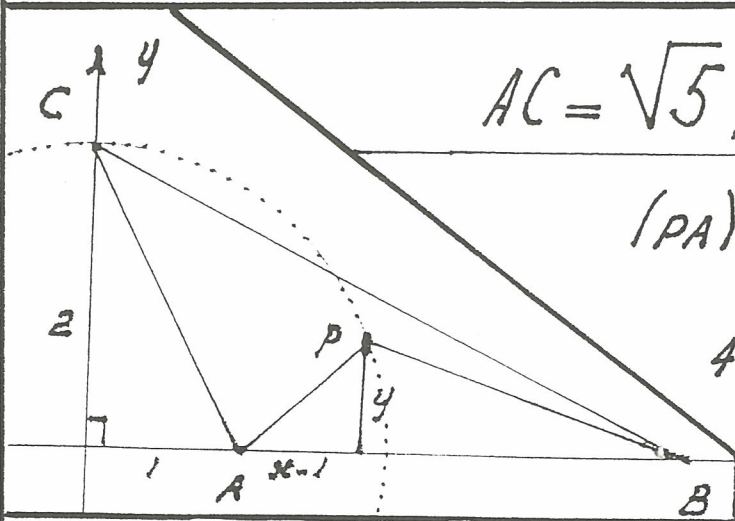
EQUATION

$$(x-1)^2 + (y-1.5)^2 = 6.25$$

EXPAND TO PROVE

$A(1, 0), B(4, 0), P(x, y). PB = 2PA$

SHOW LOCUS OF P IS A CIRCLE. PROVE $BC = 2AC$ (WHEN $C(0, 2)$)



$$AC = \sqrt{5}, BC = 2\sqrt{5} \text{ q.e.d.}$$

$$(PA)^2 = (x-1)^2 + y^2 = x^2 - 2x + 1 + y^2$$

$$4(PA)^2 = 4x^2 - 8x + 4 + 4y^2 \quad (1)$$

$$(PB)^2 = 16 - 8x + x^2 + y^2 \quad (2)$$

$$(1) = (2) \rightarrow x^2 + y^2 = 4$$

LOCUS OF P IS A CIRCLE, CENTRE $(0, 0)$, $R = 2$
 C LIES ON THAT CIRCLE

CIRCLES

18.

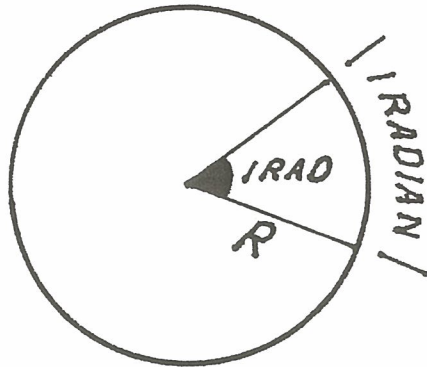
CONVERSIONS

DEGREES ↔ RADIANS

$$180^\circ = \pi \text{ RAD.}$$

$$6^\circ = \frac{6\pi^\circ}{180}$$

$$\text{OR } .1^\circ$$



LINEAR
MEASURE

$$\text{ONE RADIANT} = R$$

$$2\pi \text{ RAD.} = 2\pi R = 360^\circ$$

SIN 1.2

BEWARE!
RADIANS

MODE 5 (CASIO)
1.2 SIN .93

$$\pi/2^\circ = 90^\circ$$

$$\pi/5^\circ = 36^\circ$$

$$\pi/3^\circ = 60^\circ$$

$$\pi/6^\circ = 30^\circ$$

$$\pi/4^\circ = 45^\circ$$

$$\pi/9^\circ = 20^\circ$$

$$\pi^\circ = 180^\circ$$

$$1 \text{ RAD.} \div 57^\circ$$

SECTOR

$$\phi = .86^\circ, R = 6 \text{ u}$$

SECTOR

$$\text{ARC} = 9 \text{ cm}, R = 8 \text{ cm}$$

SECTOR

$$\phi = \pi/6^\circ, A = 4.7 \text{ cm}^2$$

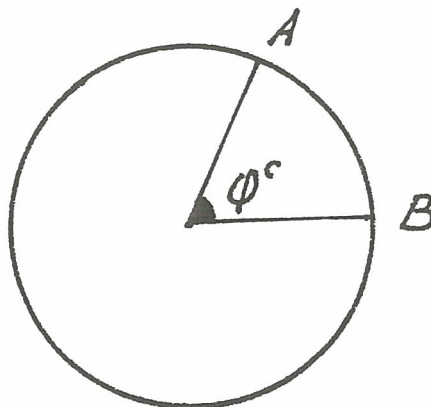
$$\text{ARC} = .86^\circ$$

$$= 5.16 \text{ u}$$

AREA SECTOR

$$15.48 \text{ u}^2$$

USING
TRIANGLE
FORMULA!



$$\text{ARC AB} = 8\phi = 9$$

$$\phi = 1.125^\circ$$

$$\text{AREA SECTOR} = 36 \text{ cm}^2$$

$$\downarrow 30^\circ$$

AREA SECTOR

$$\frac{1}{12} \pi R^2 = 4.7$$

$$R \div 4.24 \text{ cm}$$

CIRCLES

19.

$P_{\text{SECTOR}} = 12 \text{ cm.}$ ^{PHI}
CENTRE ANGLE = ϕ RAD.

SHOW $A_{\text{MAX}} = 9 \text{ cm}^2$

THE STRATEGIES

1. THINK

THE AREA OF A SECTOR IS THE AREASUM OF AN INFINITE NUMBER OF TRIANGLES: TOTAL BASE $\phi R \text{ cm}$
HEIGHT $R \text{ cm}$

THE DETAILS

2. THINK & DIFFERENTIATION WITH RESPECT TO ϕ , THE VARIABLE

$$A = \frac{1}{2} R^2 \phi \quad (1)$$

FROM DATA: $R(2+\phi) = 12$
SUB (1)

$A' = 0$, FIND ϕ & SUB. IN A
3. EXPRESS R IN TERMS OF ϕ

$$A = \frac{72\phi}{(2+\phi)^2}$$

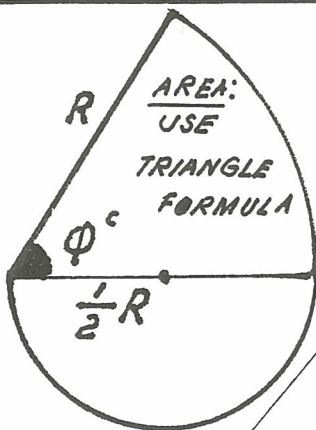
$$A' = \frac{(2+\phi)^2 - 2\phi(2+\phi)}{(2+\phi)^4} = 0$$

$$\phi = 2 \quad (\phi \neq -2)$$

$$A_{\text{MAX}} = 9 \text{ cm}^2 \text{ q.e.d.}$$

QUOTIENT RULE

EXPRESS P IN TERMS OF R & ϕ . $A = 1 \text{ U}^2$
SHOW $P = \frac{2}{R} + R(1 + \frac{\pi}{4})$. LEAST WHEN $R^2 = \frac{8}{\pi+4}$. ϕ NEAREST DEGREE



$$P = R + \phi R + \frac{1}{2} \pi R \quad (1)$$

$$A = \frac{1}{2} R^2 \phi + \frac{1}{8} \pi R^2 = 1$$

$$\phi = \frac{2}{R^2} - \frac{\pi}{4} \text{ SUB (1) } P \text{ AS SHOWN}$$

$$P' = \frac{-2}{R^3} + \frac{4+\pi}{4}, \quad P'' = \frac{4}{R^3} \quad (70)$$

MINIMUM P WHEN $P' = 0$, WHEN $R^2 = \frac{8}{\pi+4}$. q.e.d.

$$\phi = \frac{\pi+4}{4} - \frac{\pi}{4} = 1^\circ \div 57^\circ$$

CURVE SKETCHING

20.

INVOLVES PLOTTING:

TURNING POINTS
STATIONARY

$$y' = 0$$

$y'' > 0$, MIN.
 $y'' < 0$, MAX

POINTS OF INFLEXION

$y'' = 0$ AND
CHANGES SIGN

y-INTERCEPTS

$$x = 0$$

x-INTERCEPTS

WHERE
POSSIBLE

$$y = 0$$

OTHER POINTS

TO CLARIFY THE GRAPH

x WELL CHOSEN

$$y = -x^3 + 3x + 4$$

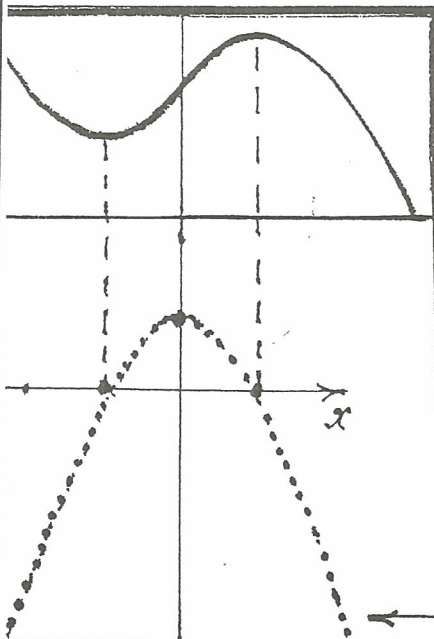
PREVIEW: NEG.

LEFT ARM

UP

CUBIC CURVE

, RIGHT ARM DOWN.



$$y' = -3x^2 + 3 = 0$$

$$x = 1$$

$$x = -1$$

$$x = 0, y' = 3$$

$$x = \pm 2, y' = -9$$

y'-CURVE

$$y'' = -6x$$

$$y'' < 0$$

$$y'' > 0$$

SIGN CHANGE

WHEN $x = 0$

TANGENT CHANGES SIDES.

IN TANGENT = y' DECREASING

MAX(1, 6)

MIN(-1, 2)

INFLEXION

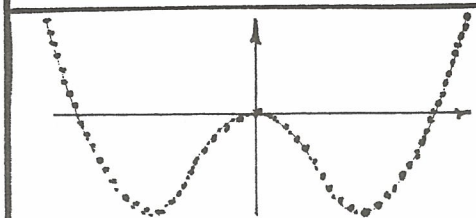
AT y-INT.

(0, 4)

CURVE SKETCHING

21.

$$y = x^4 - 2x^2$$



EVEN FUNCTION

$$x = \pm 1$$

$$y = -1$$

x -INT. $(-\sqrt{2}, 0), (0, 0), (\sqrt{2}, 0)$

y -INT. $(0, 0)$

$$y' = 4x^3 - 4x = 0$$

$$y'' = 12x^2 - 4$$

$$x = 0$$

$$x = -1$$

$$x = 1$$

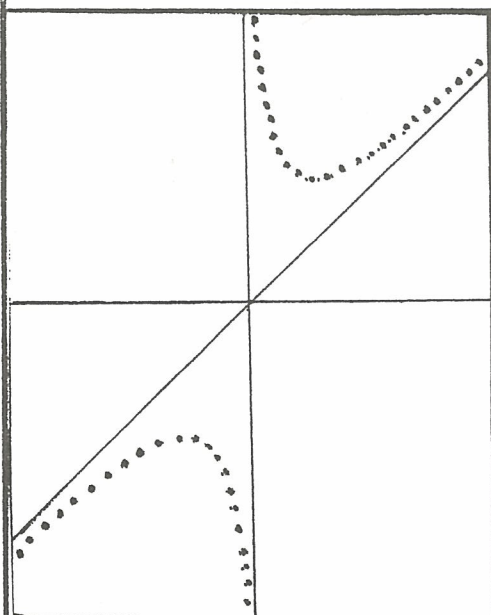
$$\left. \begin{array}{l} y'' < 0 \\ y'' > 0 \\ y'' > 0 \end{array} \right\} \begin{array}{l} \text{SIGN} \\ \text{CHANGE} \end{array}$$

MAX. $(0, 0)$
MIN. $(-1, -1)$
MIN. $(1, -1)$

$$12x^2 - 4 = 0, x = \pm \frac{1}{\sqrt{3}}$$

INFLECTIONS
 $(\frac{1}{\sqrt{3}}, -\frac{5}{9}), (-\frac{1}{\sqrt{3}}, -\frac{5}{9})$

$$y = x + \frac{4}{x}$$



ODD FUNCTION

$$x = 4, y = 5$$

$$x = -4, y = -5$$

$\lim_{x \rightarrow \infty} \frac{4}{x} = 0 \therefore$
 $x \rightarrow \infty$, ASYMPTOTE $y = x$
 $x \neq 0$, y -AXIS ASYMPTOTE
NOT TOGETHER

$$y' = 1 - \frac{4}{x^2} = 0$$

$$x = 2$$

$$x = -2$$

$$y'' > 0$$

$$y'' < 0$$

$$y'' = \frac{8}{x^3}$$

MINIMUM AT $(2, 4)$. MAXIMUM AT $(-2, -4)$

CURVE SKETCHING

22.

ALL PASS THROUGH (0,1)

EXPONENTIAL FUNCTIONS

INDEX

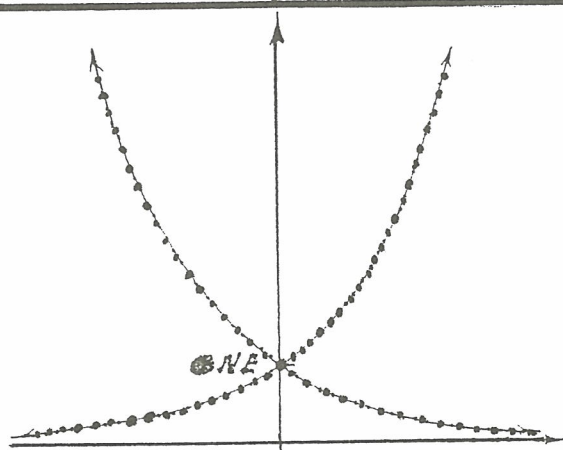
$$y = 2^{-x}$$

USE
 x^y (CASIO)

$$y = 2^x$$

$$y = 7^x$$

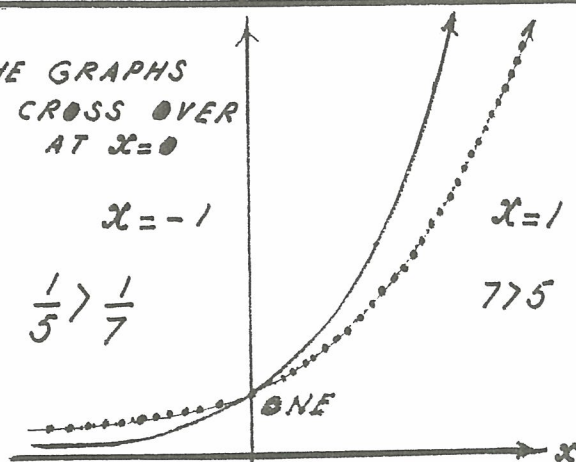
$$y = 5^x$$



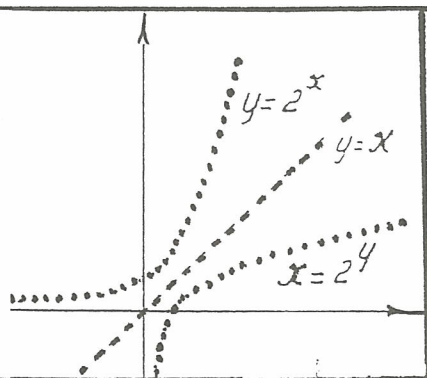
THE GRAPHS
CROSS OVER
AT $x=0$

$$x = -1$$

$$\frac{1}{5} > \frac{1}{7}$$



INVERSE EXPONENTIAL = LOGARITHMIC



THE INVERSE OF

$$y = 2^x \text{ IS } x = 2^y$$

WHICH MEANS

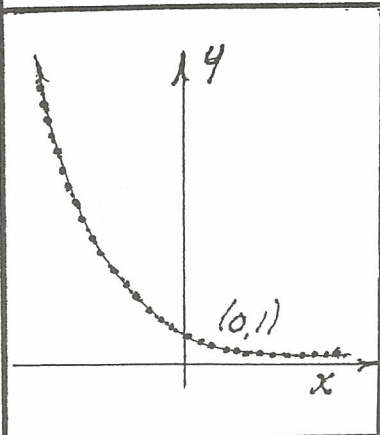
$$y = \log_2 x$$

IT IS THE REFLEXION OF $y = 2^x$
IN THE LINE $y = x$

CURVE SKETCHING

23

$$y = e^{-x}$$



USE e^x (CASIO)

TO FIND THE VALUE OF e ,

PRESS 1 INV. $\ln$

2.718 TO REMEMBER
(3×9) $\times$ (2×9)

ALL EXPONENTIAL FUNCTIONS PASS THROUGH $(0, 1)$

CALCULATOR VALUES ARE PLOTTED IMMEDIATELY,
(NOT WRITTEN)

PIECEMEAL FUNCTIONS

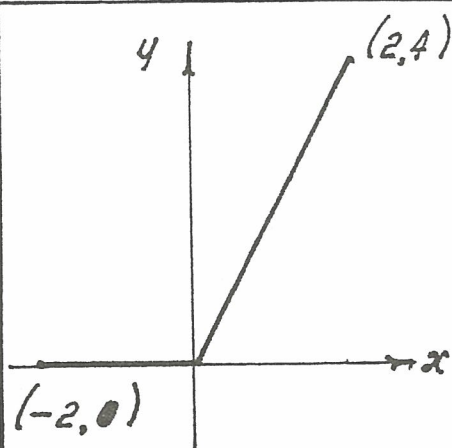
$$-2 \leq x \leq 2$$

$$f(x) = \begin{cases} 0 & \text{FOR } x \leq 0 \\ 2x & \text{FOR } x > 0 \end{cases}$$

$$f(x) = \begin{cases} |x| & \text{FOR } x \leq 1 \\ x^3 & \text{FOR } 1 < x \leq 3 \\ 36 - x^2 & \text{FOR } x > 3 \end{cases}$$

$$y = -1 \text{ FOR } x \leq 1$$

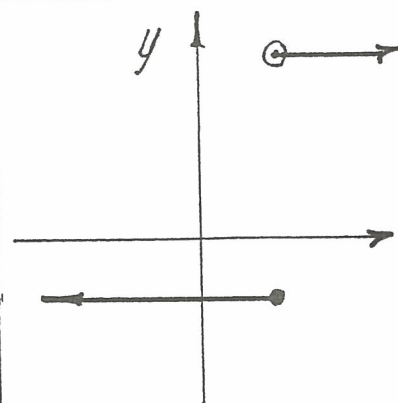
$$y = 3 \text{ FOR } x > 1$$



$$f(-2) = 2$$

$$f(3) = 27$$

$$f(5) = 11$$



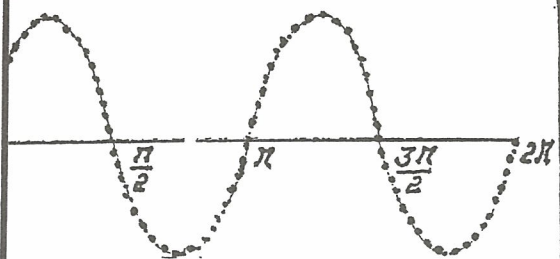
CURVE SKETCHING

24.

HALF TABLE
HALF GRAPH

$$y = \sin 2x$$

DOMAIN
 $-2\pi \leq x \leq 2\pi$



PERIOD $\frac{2\pi}{2}$

AMPLITUDE 1
RANGE $-1 \leq y \leq 1$

USE $\bullet$ AND MULTIPLES OF HALF π
USE A CONVENIENT SCALE.

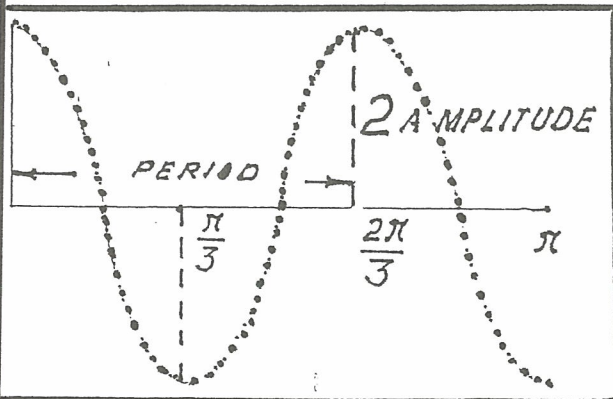
VISUALISE THE "SIN-COS CLOCK" (UNIT-CIRCLE)

THE BACK TO FRONT METHOD

$2x$	$\bullet$	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$	4π
$\sin 2x$	$\bullet$	1	$\bullet$	-1	$\bullet$	1	$\bullet$	-1	$\bullet$
DOMAIN x	$\bullet$	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π

$$y = 2 \cos 3x$$

$0 \leq x \leq \pi$



PERIOD $\frac{2\pi}{3}$

3 TIMES AS FAST AS x
HENCE $\frac{1}{3}$ OF ITS TIME

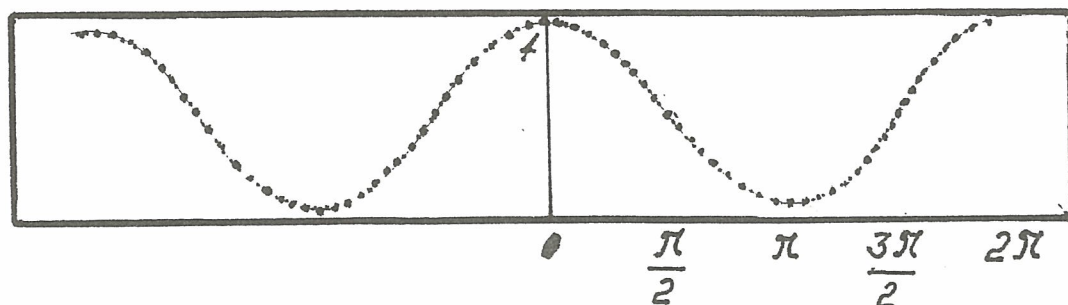
$3x$	$\bullet$	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$\cos 3x$	1	$\bullet$	-1	$\bullet$	1	$\bullet$	-1
$2 \cos 3x$	2	$\bullet$	-2	$\bullet$	2	$\bullet$	-2
x	$\bullet$	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π

CURVE SKETCHING

25.

HALF TABLE	$y = 2 + 2\cos x$ DOMAIN $-2\pi \leq x \leq 2\pi$				
x	0	$\pi/2$	π	$3\pi/2$	2π
$\cos x$	1	0	-1	0	1
$2\cos x$	2	0	-2	0	2
$2+2\cos x$	4	2	0	2	4

RANGE $-4 \leq y \leq 4$



HALF TABLE	$y = 3\cos \frac{1}{2}x$ $-\pi \leq x \leq \pi$		
$\frac{1}{2}x$	0	$\pi/2$	CALCULATION ONLY
$\cos \frac{1}{2}x$	1	0	
$3\cos \frac{1}{2}x$	3	0	PLOTING POINTS
x	0	π	

CHECK WITH		CALCULUS
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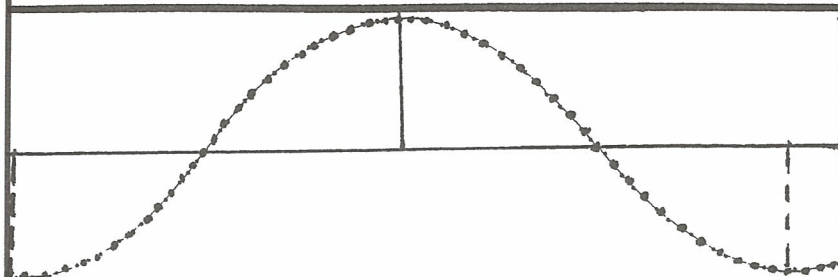
$y' = -\frac{3}{2}\sin \frac{1}{2}x = 0$ $x = 0$	$y'' = -\frac{3}{4}\cos \frac{1}{2}x$ $y'' < 0, \text{MAX}(0, 3)$	x-INTERCEPTS	
		$\cos \frac{1}{2}x = 0$	
		$(-\pi, 0)$	$(\pi, 0)$

CURVE SKETCHING

26.

$$y = \cos \pi x$$

DOMAIN
 $-1 \leq x \leq 1$



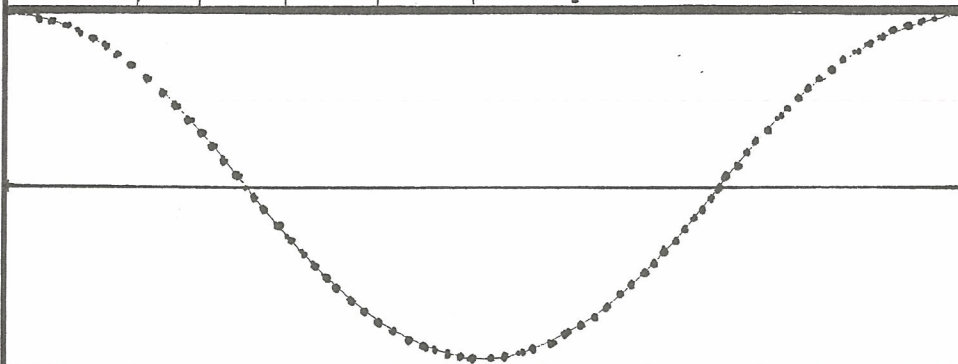
$y' = -\pi \sin \pi x = 0$	$y'' = -\pi^2 \cos \pi x$
$x = 0$	$y'' < 0$
$x = \pm 1$	$y'' > 0$

MAXIMUM AT $(0, 1)$. MINIMA AT $(1, -1)$ & $(-1, -1)$

$$y = \cos 2x$$

DOMAIN
 $0 \leq x \leq 180^\circ$

$2x$	0	90	180	270	360	VALUES OF x FOR WHICH $\cos 2x = \frac{1}{2}$	
$\cos 2x$	1	0	-1	0	1	$2x = 60$	$2x = 300$
x	0	45	90	135	180	$x = 30^\circ$	$x = 150^\circ$



$y' = -2 \sin 2x = 0$
 $y'' = -4 \cos 2x$
 $2x = 0, 360^\circ \text{ \& } 180^\circ$
 MAX. $(0, 1)$ & $(\pi, 1)$
 MIN. $(\frac{\pi}{2}, -1)$

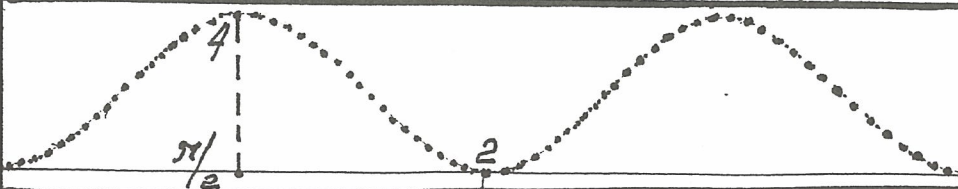
POSITION
MOVING
PARTICLE

$$x = 3 - \cos 2t$$

HALF TABLE

$0 \leq t \leq 2\pi$
 AMPLITUDE 2
 PERIOD π

$2t$	0	$\pi/2$	π	$3\pi/2$	2π	INITIALLY ($t=0$) THE PARTICLE IS AT $x=2$
$\cos 2t$	1	0	-1	0	1	
$3 - \cos 2t$	2	3	4	3	2	
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	



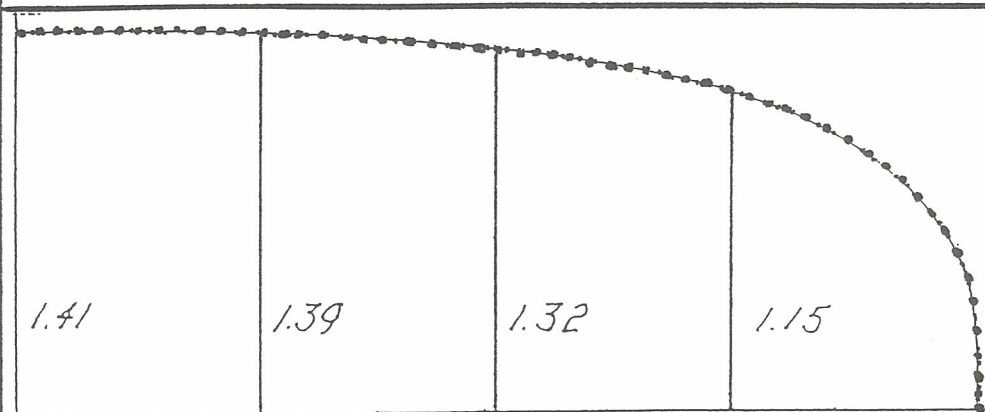
NOTE: THE LOCUS OF THE PARTICLE IS A STRAIGHT LINE. THE PARTICLE DOES NOT FOLLOW THE CURVE WHICH IS A FUNCTION OF TIME

CURVE SKETCHING

27.

$$y = (4 - x^2)^{1/4}$$

$$x: 0, \frac{1}{2}, 1, \frac{1}{2}, 2$$



AREA UNDER
CURVE

$$\frac{1}{4} (1.41 + 2 \times 3.86) \div 2.28 \text{ U}^2$$

TRAPEZIUM
RULE

SIMPSON $\frac{1}{6} \{ 1.41 + 2 \times 1.32 + 4(1.39 + 1.15) \} \div 2.37 \text{ U}^2$

$$y = (x-1)(x^2-5)$$

DOMAIN

$$-3 \leq x \leq 3$$

$$y = x^3 - x^2 - 5x + 5$$

$$y' = 3x^2 - 2x - 5 = 0$$

$$(3x-5)(x+1) = 0$$

$$y'' = 6x - 2 = 0$$

$$x = -1$$

$$x = 5/3$$

$$y'' < 0$$

$$y'' > 0$$

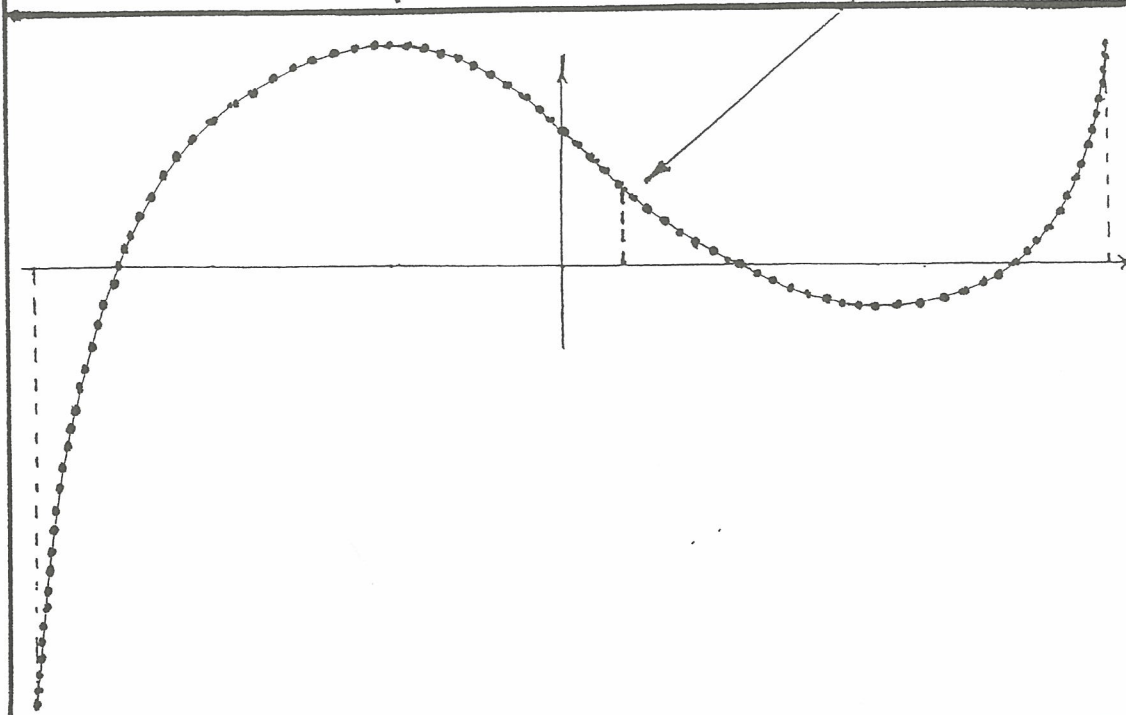
SIGN
CHANGE
AT $x = 1/3$

INFLEXION

$$\left(\frac{1}{3}, \frac{88}{27} \right)$$

MAX $(-1, 8)$

MIN $(5/3, -40/27)$



x-INT.

$$(1, 0)$$

$$(\sqrt{5}, 0)$$

$$(-\sqrt{5}, 0)$$

END VALUES

$$-16$$

$$8$$

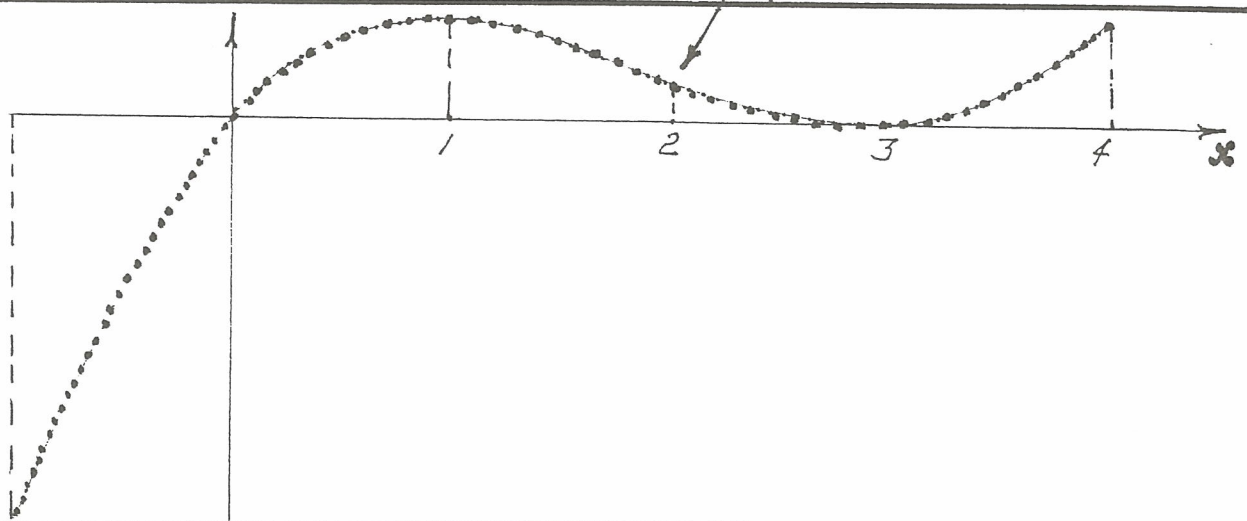
CURVE SKETCHING

28.

$$y = x(x-3)^2$$

$$-1 \leq x \leq 4$$

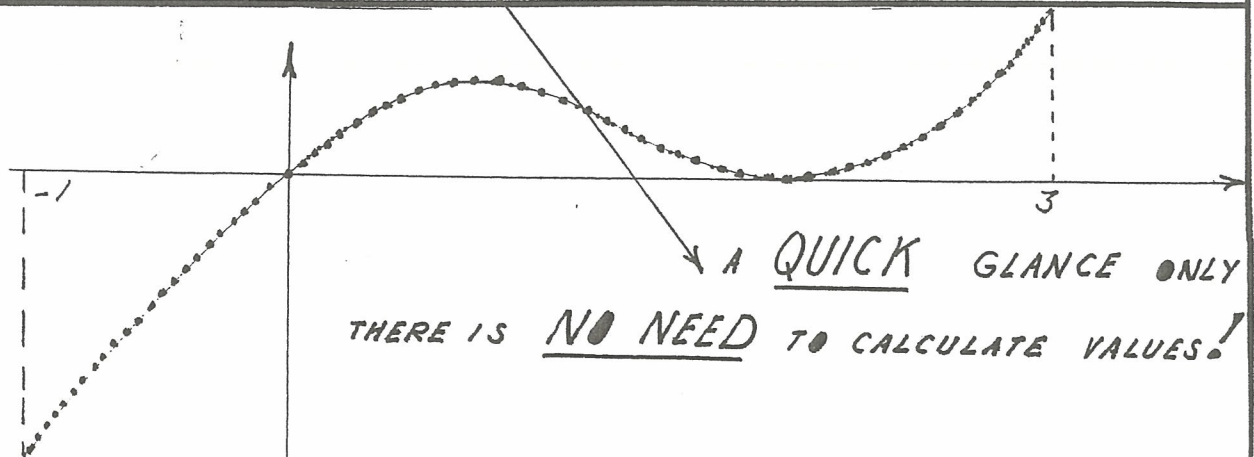
END VALUES -16 & 4	y - INTERCEPT (0,0)	x - INTERCEPTS (0,0) & (3,0)
$y' = 3x^2 - 12x + 9 = 0$ $(x-3)(x-1) = 0$ $x = 3$ $x = 1$	$y'' = 6x - 12$ $y'' > 0$ $y'' < 0$	SIGN CHANGE $x = 2$ INFL. AT (2,2) MINIMUM AT (3,0) MAXIMUM AT (1,4)



$$y = 9x(x-2)^2$$

$$-1 \leq x \leq 3$$

$y' = 27x^2 - 72x + 36 = 0$ $(3x-2)(x-2) = 0$, $x = 2$ $x = 2/3$	$y'' = 54x - 72 = 0$ $x = 2$ $x = 2/3$ $y'' > 0$ $y < 0$	INFLEXION AT $(4/3, 16/3)$ SIGN CHANGE WHEN $x = 4/3$ MINIMUM AT (2,0) MAXIMUM AT $(2/3, 32/3)$
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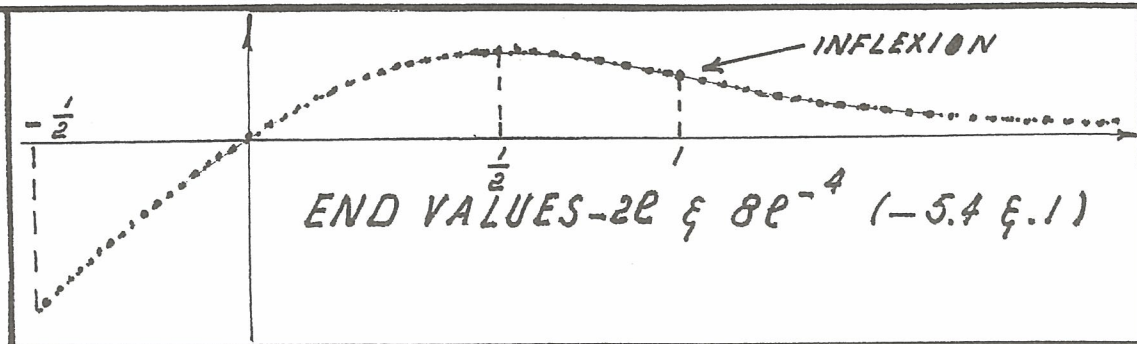


A QUICK GLANCE ONLY
THERE IS NO NEED TO CALCULATE VALUES!

CURVE SKETCHING

29.

$$y = 4xe^{-2x} \quad -\frac{1}{2} \leq x \leq 2$$



THINK $y' = \frac{\text{PRODUCT RULE}}{y'y + y'y'}$ INCL. FUNCTION OF A FUNCTION
DER. PART X DER. WHOLE

$$y' = 4e^{-2x} - 8xe^{-2x} = 0$$

$$\underbrace{e^{-2x}}_{\text{ALWAYS } > 0} (4 - 8x) = 0$$

$$x = \frac{1}{2}$$

$$y'' = -8e^{-2x} - 8e^{-2x} + 16xe^{-2x}$$

SUB $x = \frac{1}{2}$ $y'' = \frac{-8}{e}$ WHICH IS < 0
MAXIMUM AT $(\frac{1}{2}, \frac{2}{e})$

$$y'' = -16e^{-2x} + 16xe^{-2x} = 0$$

$$\underbrace{16e^{-2x}}_{\text{ALWAYS } > 0} (x-1) = 0$$

$x = 1$

CHECK

$x = \frac{1}{2}, y'' < 0$

$x = 2, y'' > 0$

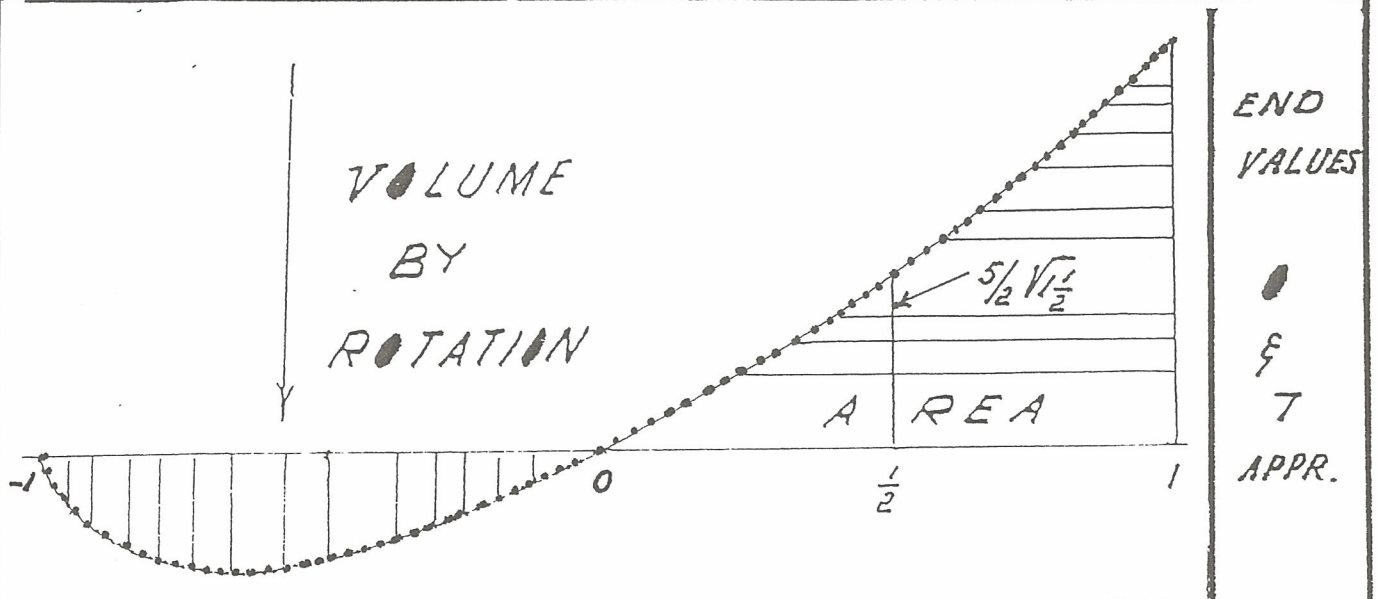
SIGN CHANGE
 y'' WHEN $x = 1$
INFLEXION AT $(1, \frac{4}{e^2})$

CURVE SKETCHING

30.

$$y = 5x\sqrt{x+1}$$

$$\text{DOMAIN } -1 \leq x \leq 1$$



$$y = 5x(x+1)^{\frac{1}{2}}$$

$$y' = 5\sqrt{x+1} + 5x \left\{ \frac{1}{2}(x+1)^{-\frac{1}{2}} \right\}$$

PRODUCT RULE

$$y'y + y'y'$$

FUNCTION OF A FUNCTION

DERIVATIVE PART

TIMES DERIVATIVE WHOLE

$$y' = 5\sqrt{x+1} + \frac{5x}{2\sqrt{x+1}} = 0$$

$$x = -\frac{2}{3}$$

RATHER THAN FINDING y'' , CHECK $x = \xi = -0.8$ ($x \neq -1$)
 $y' = 5\xi - 2.2$ RESPECTIVELY AND THUS $(-\frac{2}{3}, y)$ IS A MIN.

WHEN $x = -1$, THE TANGENT IS VERTICAL.

VOLUME

$$y^2 = 25(x^3 + x^2)$$

$$V = 25\pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 = \frac{25\pi}{12} u^3$$

AREA WITH SIMPSON

2 INTER VALS

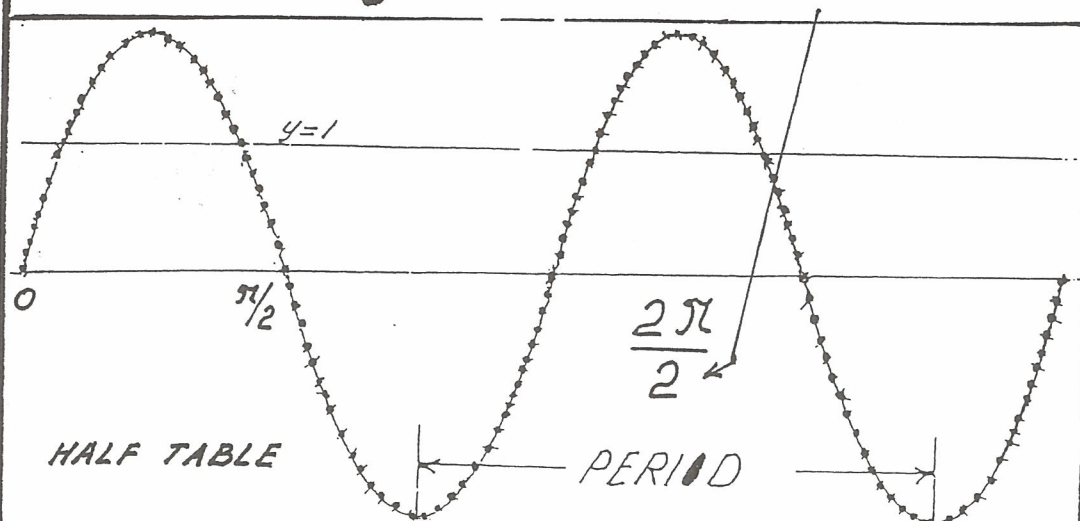
$$\frac{1}{3} (\text{FIRST} + \text{LAST} + 4 \times \text{MIDDLE})$$

$$A = \frac{1}{6} (7 + 10\sqrt{1.5}) \div 3.2 u^2$$

CURVE SKETCHING

31

$$y = 2 \sin 2x \quad \text{DOMAIN} \quad 0 \leq x \leq 2\pi$$



BACK
TO
FRONT
METHOD

$2x$		$\pi/2$	π	$\frac{3\pi}{2}$	2π
$\sin 2x$	0	1	0	-1	0
$2 \sin 2x$	0	2	0	-2	0
x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π

RANGE - $2 \leq y \leq 2$

AMPLITUDE 2

PERIOD π

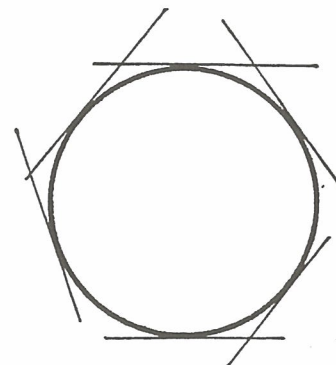
VALUES OF x IN THIS DOMAIN FOR WHICH
 $y \geq 1$ (ABOVE/ON $y=1$)

$2 \sin 2x \geq 1$	$\sin 2x \geq \frac{1}{2}$	BASIC ANGLE $\frac{\pi}{6} = 30^\circ$
$\frac{\pi}{6} \leq 2x \leq \frac{5\pi}{6}$		$\frac{13\pi}{6} \leq 2x \leq \frac{17\pi}{6}$
$\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$		$\frac{13\pi}{12} \leq x \leq \frac{17\pi}{12}$
		$\frac{25\pi}{12} \leq \frac{29\pi}{12}$ OUTSIDE THE DOMAIN

DIFFERENTIATION

32

CALCULATING DIFFERENT (CHANGING)
GRADIENTS
OF THE TANGENTS
TO A CURVE



THE PROCESS OF FINDING m TANGENT OR y' OR
 $f'(x)$ OR THE DIFFERENTIAL OR THE GRADIENT FUNCTION
OR $\frac{dy}{dx}$ OR $\frac{d}{dx}$ OR THE DERIVATIVE, IS CALLED
DERIVATION BECAUSE
 y' IS DERIVED FROM $y = f(x)$

VISUALISE THE ROUTINE

$$y = 2x^5 \quad \begin{array}{l} \text{MULTIPLY} \\ \text{TAKE ONE} \end{array}$$

$m_{\text{TAN}} = y' = 10x^4$
= 10 (84°) AT (1,2)
THE
FIRST DERIVATIVE
GIVES THE CHANGE
IN CURVATURE

THE SECOND
DERIVATIVE

GIVES THE CHANGE OF
THE FIRST

$$y'' = 40x^3$$

DIFFERENTIATION

33

THE GRADIENT OF THE WHOLE
IS THE GRADIENT SUM OF THE PARTS

$$y = \underbrace{3x^3}_{\text{CUBIC CURVE}} - \underbrace{5x^2}_{\text{+ PARABOLA}} + \underbrace{5x}_{\text{+ STRAIGHT LINE}} + \underbrace{6}_{\text{+ STRAIGHT LINE } y=6, m=0}$$

$$m_{TAN} = y' = 9x^2$$

+ PARABOLA

$$m_{TAN} = y' = -10x$$

+ STRAIGHT LINE

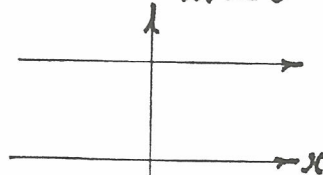
$$m = y' = 5$$

INV. TAN: 79° (CASIO)

INCLINATION MEASURED
FROM POS. X-AXIS

+ STRAIGHT LINE $y = 6$

$$m = 0$$



TOTAL

$$y' = 9x^2 - 10x + 5$$

AT
(0,6)

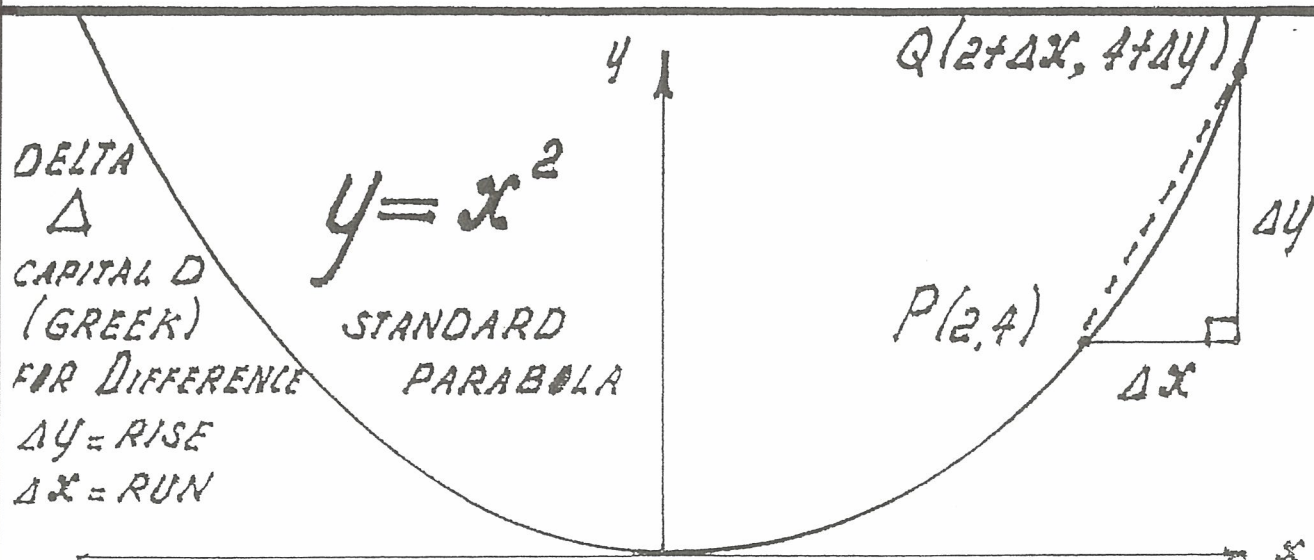
$$y' = 5$$

DIFFERENTIATION

34.

FROM FIRST PRINCIPLES

MEANS: PROVING THE ROUTINE (FROM SCRATCH)



AVERAGE GRADIENT $PQ = \frac{\Delta y}{\Delta x}$

Q LIES ON THE PARABOLA, SO ITS COORDINATES SATISFY $y = x^2$

$$(4 + \Delta y) = 4 + 4\Delta x + \Delta x^2$$

$$\frac{\Delta y}{\Delta x} = 4 + \Delta x$$

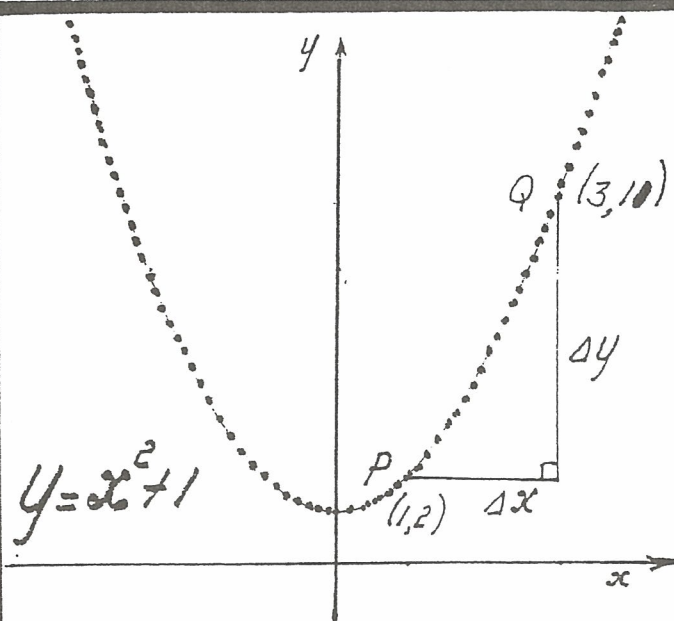
$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{d}{dx} = y' = m_{\text{TANGENT AT } P} = 4$$

$\Delta x \rightarrow 0$
APPROACHES

FROM SECANT TO TANGENT

ROUTINE

$y' = 2x$, SUB 2, $= 4$
q.e.d.



AVERAGE GRADIENT
BETWEEN P & Q

GRADIENT SECANT

$$\frac{\Delta y}{\Delta x} = \frac{\text{delta } y}{\text{delta } x} = \frac{8}{2} = 4$$

$\Delta = D$ IN GREEK } FOR DIFFERENCE
 $\delta = d$

GRADIENT AT P FROM FIRST PRINCIPLES
PROOF OF THE ROUTINE

LEARN THE FOLLOWING
ALL PURPOSE PROCEDURE

1. START FROM $Q(x + \Delta x, y + \Delta y)$ RANDOM POINT

2. SUBSTITUTE Q

3. EXPAND

4. SUBTRACT

5. DIVIDE BY Δx

6. LET Δx APPROACH 0

INCLINATION
ANGLE WITH POS. x -axis

$$y = x^2 + 1 \quad y' = 2x$$

$$y + \Delta y = (x + \Delta x)^2 + 1$$

$$y + \Delta y = x^2 + 2x \cdot \Delta x + \Delta x^2 + 1$$

$$y = x^2 + 1$$

$$\Delta y = 2x \cdot \Delta x + \Delta x^2$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y' = m \text{ TAN AT P} = 2x$$

SINCE $x = 1$, $y' = 2 = m \text{ TANGENT}$
INV. TAN 63°

DIFFERENTIATION

36.

$$y = 3x^2 - 5 \quad \text{DIFFERENTIATE FROM}$$

FIRST PRINCIPLES. FIND m_{TAN} WHERE $x = -1$ & $x = 3$

1. USE $P(x, y)$ & $Q(x + \Delta x, y + \Delta y)$

2. SUBSTITUTE Q

$$y + \Delta y = 3(x + \Delta x)^2 - 5$$

3. EXPAND

$$y + \Delta y = 3x^2 + 6x \cdot \Delta x + 3\Delta x^2 - 5$$

4. SUBTRACT

$$y = 3x^2 - 5$$

$$\Delta y = 6x \cdot \Delta x + 3\Delta x^2$$

5. DIVIDE BY Δx

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x$$

6. LET Δx APPROACH 0
Q APPROACHES P

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y' = m_{\text{TANGENT}} = 6x$$

$$y = 3x^2 - 5$$

$$y' = 6x \quad \text{ROUTINE PROCEDURE GRADIENT FUNCTION}$$

WHEN $x = -1$

OR DIFFERENTIAL COEFFICIENT

AT $(-1, -2)$, $y' = -6$

WHEN $x = 3$

AT $(3, 22)$, $y' = 18$

DIFFERENTIATION

37.

$$y = 4x^2 - 2x + 6 \quad (1)$$

FIND $P(x, y)$ WHERE THE TANGENT IS PARALLEL TO x -AXIS. FROM FIRST PRINCIPLES

1. USE $Q(x + \Delta x, y + \Delta y)$

LEARN THE PROCEDURE,
PRACTISE THE DETAILS.

2. SUBSTITUTE Q

$$y + \Delta y = 4(x + \Delta x)^2 - 2(x + \Delta x) + 6$$

3. EXPAND

$$y + \Delta y = 4x^2 + 8x\Delta x + 4\Delta x^2 - 2x - 2\Delta x + 6$$

4. SUBTRACT (1)

$$y = 4x^2 - 2x + 6$$

$$\Delta y = 8x\Delta x + 4\Delta x^2 - 2\Delta x$$

5. DIVIDE BY Δx

$$\frac{\Delta y}{\Delta x} = 8x + 4\Delta x - 2$$

6. LET Δx APPROACH 0

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y' = m \text{ TANGENT} = 8x - 2$$

TAKE 1

$$y = 4x^2 - 2x + 6$$

$$y' = 8x - 2$$

ROUTINE
PROCEDURE

TANGENT $\parallel$ x -AXIS

WHEN

$$8x - 2 = 0 \quad \text{OR} \quad x = \frac{1}{4},$$

WHICH IS AT $P\left(\frac{1}{4}, 5\frac{3}{4}\right)$

DIFFERENTIATION

38.

$$y = (x-2)(2x+3)$$

FIRST SECOND
PRODUCT RULE

A LOGICAL AND SENSIBLE
START TOPIC
= FIRST = DIFF.

VARIATION

REMEMBER:
MULTIPL. IS SPEEDY ADDITION

VISUALISE

$$y'y + yy'$$

SAY
(VERBAL REHEARSAL)

THE DERIVATIVE OF THE FIRST
TIMES THE ORDINARY SECOND
PLUS
THE ORDINARY FIRST

TIMES THE DERIVATIVE OF THE SECOND

$$y' = (2x+3) + (x-2) \times 2$$

$$\rightarrow y' = \boxed{4x-1}, \text{ GRADIENT FUNCTION}$$

IT GIVES THE CHANGE IN CURVATURE

ROUTINE CHECK

$$y = 2x^2 - x - 6$$

$$\rightarrow y' = \boxed{4x-1}$$

DIFFERENTIATION

39.

$$y = \frac{\overset{\text{FIRST}}{\cos x}}{\underset{\text{SECOND}}{1 + \sin x}}$$

QUOTIENT
RULE

VISUALISE

REMEMBER,

DIVISION IS SPEEDY SUBTR.

$$\frac{yy - yy}{\text{SQUARED}}$$

LEAVE OUT x FOR CONVIENCE!

QUICKER & LESS CUMBERSOME.
FORGET ABOUT PROTOCOL.

$$y' = \frac{-\sin(1 + \sin) - \cos^2}{(1 + \sin)^2}$$

SIMPLIFY

USE THE IDENTITY $\sin^2 + \cos^2 = 1$

FINAL
ANSWER

$$y' = \frac{-1}{(1 + \sin x)}$$

$\sin x \neq -1$
 $x \neq -\pi/2$

REMEMBER

$$y = \sin$$

$$y' = \text{POS } \cos$$

REMEMBER

$$y = \cos$$

$$y' = \text{NEG } \sin$$

REMEMBER

$$y = \tan$$

$$y' = \sec^2$$

DIFFERENTIATION

40.

DIFFERENTIATE WITH RESPECT TO x

$$y = \frac{3}{x^2} + \frac{4}{x}$$

EQUATIONS

TANGENT &
NORMAL
(PERPENDICULAR)
AT (1,7)

ROUTINE: ADD ONE

$$y' = \frac{-6}{x^3} - \frac{4}{x^2} = -10$$

AT (1,7)

VISUALISE
ONLY

$$y = mx + b$$

$$7 = -10 + b$$

WE DON'T NEED
b HERE!

EQUATION

$$y = -10x + 17$$

TANGENT

EQUATION

$$y = \frac{1}{10}x + \frac{69}{10}$$

NORMAL

THINK: OPPOSITE, RECIPROCAL

DIFFERENTIATION

41.

$$y = \cos 3x$$

FUNCTION OF A FUNCTION
COMPOSITE FUNCTION: 2 IN ONE
(RUSSIAN DOLL)

CHAIN RULE

ROUTINE
IN WORDS!

SAY AND REMEMBER

DERIVATIVE
TIMES
DERIVATIVE

PART

WHOLE

$$y' = -3 \sin 3x$$

NO UNNECESSARY WORKING!

HANDY TO KNOW WHEN YOU DO 4

UNIT
MATHS

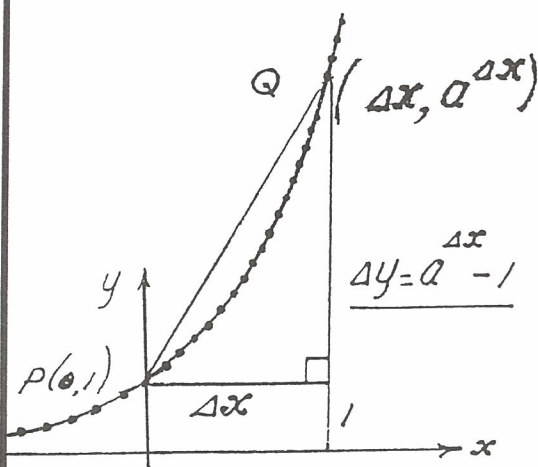
$$y = \sin$$
$$y' = \text{POS COS}$$

$$y = \cos$$
$$y' = \text{NEG SIN}$$

$$y = \tan$$
$$y' = \text{SEC}^2$$

DIFFERENTIATION

42.



$y = a^x$ EXPONENTIAL FUNCTION

EXPONENT = LOGARITHM = INDEX = POWER

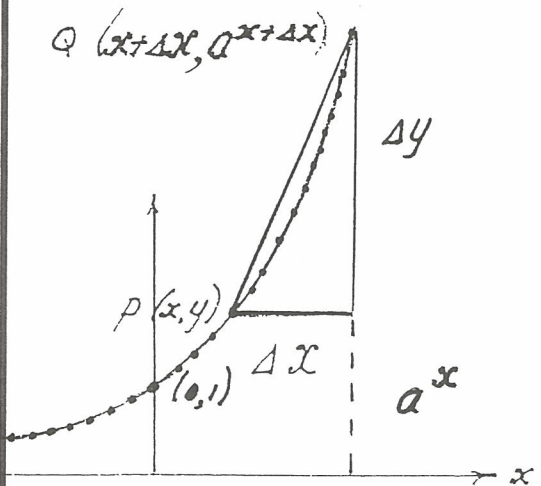
ALWAYS THROUGH $(0, 1)$

ALWAYS POSITIVE

THE GRADIENT OF THE TANGENT

FROM DIAGRAM

AT $P(0, 1)$ IS $m_0 = \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}$



GRADIENT AT $P(x, y)$

FROM FIRST PRINCIPLES

USING THE DIAGRAM

$$\Delta y = a^{x + \Delta x} - a^x$$

$$\Delta y = a^x (a^{\Delta x} - 1)$$

$$y' = \lim_{\Delta x \rightarrow 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x} = a^x \lim_{\Delta x \rightarrow 0} \frac{(a^{\Delta x} - 1)}{\Delta x} = m_0 a^x \leftarrow$$

$$y = a^x$$

$$y' = m_0 a^x = m_0 y$$

GRADIENT AT $(0, 1)$

DIFFERENTIATION

43.

FINDING THE GRADIENTS
AT (0,1) FOR 2^x AND 3^x

Δx	$\frac{2^{\Delta x} - 1}{\Delta x}$	$\frac{3^{\Delta x} - 1}{\Delta x}$
.1	.72	1.16
.01	.6955	1.105
.001	.6933	1.0992
.0001	.6931	1.0987
LIMITS	.69	1.1
FUNCTION	$y = 2^x$	$y = 3^x$
y' GRADIENT OR DERIVATIVE OR $\frac{dy}{dx}$	$m. 2^x$ $.69 \times 2^x$	$m. 3^x$ 1.1×3^x

DIFFERENTIATION

44.

A NEW NUMBER e

π IS A BIT MORE THAN 3,
WHEREAS e IS A BIT LESS THAN 3.

BOTH HAVE BEEN INTRODUCED FOR CONVENIENCE

IF $y = e^x$

$y' = e^x$

NO CHANGE!

THAT MEANS WE INTRODUCED $m_0 = 1$

THAT MEANS $\lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$

LET $\Delta x = .0001$

$e^{.0001} = 1.0001$

$\log_{10} e = \frac{\log 1.0001}{.0001}$

$= \dots$

INV. LOG $\longrightarrow$

$e \doteq 2.718$

3 (NINES)

2 (NINES) !

DIFFERENTIATION

45

$$\frac{2^{.0001} - 1}{.0001} = .69$$

$m.$

$$\frac{3^{.0001} - 1}{.0001} = 1.1$$

$m.$

$$\ln 2 = \frac{\ln 1.000169}{.0001}$$

$\log_e 2$

= ALSO .69

$$\ln 3 = \frac{\ln 1.00011}{.0001}$$

$\log_e 3$

= ALSO 1.1

AS A RESULT:

IF $y = 2^x$
INSTEAD OF

$y' = \ln 2 \cdot 2^x$
 $m. 2^x$

AS A RESULT:

IF $y = 3^x$
INSTEAD OF

$y' = \ln 3 \cdot 3^x$
 $m. 3^x$

IN GENERAL

IF $y = a^x$

$y' = \ln a \cdot a^x$

PROOF

MEANS $a = e^y$
 $y = \ln a$
 $a^x = e^{\ln a \cdot x}$
FUNCTION
OF A FUNCTION

THE DERIVATIVE
OF $e^{\ln a \cdot x} = \ln a \cdot e^{\ln a \cdot x}$
DER. PART X DER. WHOLE

* SUBSTITUTE

$\frac{da^x}{dx} = \ln a \cdot a^x$

$$y = e^{6x+3}$$

$$y = e^{2x^3}$$

TWO IN ONE
FUNCTION OF A FUNCTION

TWO IN ONE
FUNCTION OF A FUNCTION

DER. PART X DER. WHOLE

DER. PART X DER. WHOLE

UNCHANGED

UNCHANGED

$$y' = 6e^{6x+3}$$

$$y' = 6x^2 e^{2x^3}$$

$$y = x^2 e^x$$

$$y = \frac{e^x + 1}{2x}$$

PRODUCT RULE

$$y'y + yy'$$

VISUALISE

QUOTIENT RULE

$$\frac{y'y - yy'}{\text{SQUARED}}$$

$$y' = e^x(x^2 + 2x)$$

$$y'' = e^x(x^2 + 4x + 2)$$

$$y' = \frac{x.e^x - e^x - 1}{2x^2}$$

DIFFERENTIATION

47

EXERCISES WITH e (CASIO)

$$e^{2.63}$$

$$2.63 e^x \quad \swarrow \text{inv. ln} \quad 13.87377$$

$$e^{-.7}$$

$$.7 \div e^x \quad .49659$$

$$4e^{-5.6}$$

$$5.6 \div e^x \times 4 = .01479$$

TO FIND e
MEANS e^1

$$1e^x \quad 2.718$$

$$e^0 = 1$$

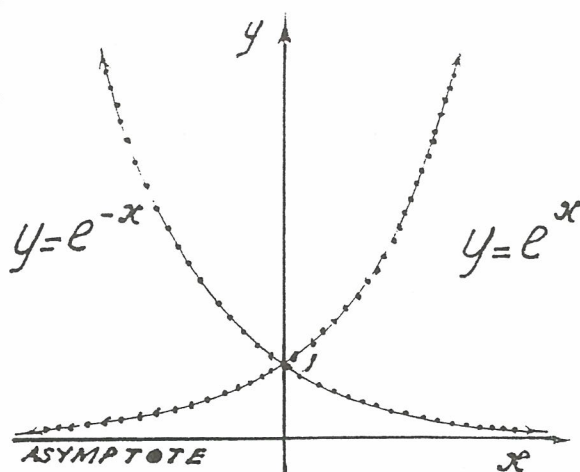
CHECK

$$0e^x \quad 1$$

$$y = e^x \quad \text{FOR } -1 \leq x \leq 3$$

x	-1	0	1	2	3
y	.36	1	2.7	7.4	20

$y = e^{-x}$ IS THE REFLECTION
OF $y = e^x$ IN THE y -AXIS



DIFFERENTIATION

48.

$$y = \ln x$$

LOGARITHM FUNCTION

THE INVERSE OF AN EXPONENTIAL ONE

↓ MEANS $x = e^y$ INVERSE OF $y = e^x$

DIFFERENTIATE
WITH RESPECT TO y

$$\frac{dx}{dy} = e^y$$

NO CHANGE

DIFFERENTIATE
WITH RESPECT TO x

$$\frac{dy}{dx} = \frac{1}{x}$$

$$y = \ln x$$

$$y' = \frac{1}{x}$$

FUNCTION OF A FUNCTION

DER. PART $\times$ DER. WHOLE

WHICH
IS
HERE

$$\frac{f'(x)}{f(x)}$$

$$y = \ln(5x-2)$$

$$y' = \frac{5}{5x-2}$$

$$y = \ln(7x^2+3x)$$

$$y' = \frac{14x+3}{7x^2+3x}$$

CHANGE OF BASE

$$y = \log_a x \quad (1)$$

$$x = a^y$$

$$\log_b x = \log_b a^y$$

$$= y \log_b a$$

SUB(1)

$$\log_b x = \log_a x \cdot \log_b a$$

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{VISUALISE}$$

AS A RESULT

$$\log_{10} x = \frac{\ln x}{\ln 10}$$

COMMON
NATURAL
NAPIERIAN
BASE e

$$\log_{10} 8.2 = \frac{\ln 8.2}{\ln 10}$$

.914 CHECK .914

$$y = \log_a x = \frac{\ln x}{\ln a \text{ (CONSTANT)}}$$

$$y' = \frac{1}{\ln a \cdot x}$$

$$y = \log_{10} x \rightarrow y' = \frac{1}{\ln 10 \cdot x}$$

$$y = \log_{10} (4x-1) \rightarrow y' = \frac{4}{\ln 10 (4x-1)}$$

DIFFERENTIATION

50

$$y = \ln 1 + \sin x$$

↑
THE WHOLE

↑
THE PART

FUNCTION OF A FUNCTION

DERIVATIVE PART $\times$ DERIVATIVE WHOLE

$$y' = \frac{\cos x}{1 + \sin x}$$

IF $y = \ln x = \log_e x$

LOGARITHMIC FUNCTION
BASE $e = 2.718$

$$y' = \frac{1}{x}$$

DIFFERENTIATION

5/

$$y = x e^{-x}$$

PRODUCT
RULE

$$y'y + yy'$$

INCLUDING
FUNCTION OF A FUNCTION

RULE

DER. PART X DER. WHOLE

BEWARE!

$$y' = e^{-x} - x e^{-x}$$

$$y = e^x$$

$$y' = e^x$$

THE SAME AS y

BUT

$$y = e^{-x}$$

$$y' = -e^{-x}$$

DER. PART = -1

DIFFERENTIATION

52.

$$y = (\cos x - \sin x)^2$$

PART y

WHOLE y

FUNCTION OF A FUNCTION

DERIVATIVE PART TIMES
DERIVATIVE WHOLE

$$y' = (-\sin - \cos) \times 2(\cos - \sin)$$

$$y' = -2(\cos^2 x - \sin^2 x)$$

$$y = \sin$$

$$y' = \text{POS COS}$$

$$y = \cos$$

$$y' = \text{NEG. SIN}$$

$$y = \tan x = \frac{\sin}{\cos}$$

$$y' = \frac{\cos^2 + \sin^2}{\cos^2}$$

$$= \frac{1}{\cos^2} = \sec^2 x$$

DIFFERENTIATION

53.

$$y = \sqrt{x^2 + 5}$$

$$y = (x^2 + 5)^{\frac{1}{2}}$$

SHORTCUT

START WITH $\frac{1}{2}$ & PICK UP

THE DER. PART ON THE WAY
DOWN! COPY BRACKETS. TAKE!

$$y' = x(x^2 + 5)^{-\frac{1}{2}}$$

$$y = \sin^2 x$$

MEANS

$$y = (\sin x)^2$$

F.F.

SHORTCUT

$$y' = 2 \cos x \sin x$$

$$y' = \sin 2x \text{ (OPTIONAL)}$$

$$y = \frac{1}{(3x+5)^2}$$

$$y = \frac{4x^3 + 5x}{x^2}$$

$$y = (3x+5)^{-2}$$

F.F.

USE SHORTCUT

$$y' = -6(3x+5)^{-3}$$

$$y = 4x + 5x^{-1}$$

$x \neq 0$

$$y' = 4 - 5x^{-2}$$

DIFFERENTIATION

54

$$y = x^{\frac{4}{3}}$$

NORMAL ROUTINE

$$y' = \frac{4}{3} x^{\frac{1}{3}}$$

$$y = \sqrt{x^5}$$

REWRITE

$$y = x^{\frac{5}{2}}$$

NORMAL ROUTINE

$$y' = \frac{5}{2} x^{\frac{3}{2}}$$

DIFFERENTIATE

$$\frac{2x-5}{x^2-3}$$

FIRST

SECOND

QUOTIENT
RULE

$$\frac{y'y - yy'}$$

SQUARED

1st end

1st

2nd

y' y

-

y

y'

$$2(x^2-3) - (2x-5)2x$$

$$(x^2-3)^2$$

$$\frac{-2(x^2-5x+3)}{(x^2-3)^2}$$

$$(x^2-3)^2$$

DIFFERENTIATION

55.

$$y = \ln(\ln x)$$

F.F. $\downarrow$ Y PART
 $\downarrow$ Y WHOLE
 DER. PART $\times$ DER. WHOLE

$$y' = \frac{1}{x \ln x}$$

$$y = \ln(x^2 - 9)$$

F.F. $\downarrow$ Y PART
 $\downarrow$ Y WHOLE

$$y' = \frac{2x}{x^2 - 9}$$

$$y = \ln(3x+1)$$

$\downarrow$ Y PART
 $\downarrow$ Y WHOLE

$$y' = \frac{3}{3x+1}$$

$$y'' = \frac{-9}{(3x+1)^2}$$

NORMAL
 ROUTINE

EQUATIONS TANGENTS

$$y = \ln x$$

WHEN $x = 1 \text{ \& } 3$

$$y' = \frac{1}{x} = 1 \text{ AT } (1, \bullet)$$

$$\bullet = 1 + b \therefore y = x - 1$$

REMEMBER: $\ln_e 1 = 0$ ($e^0 = 1$)

$$y' = \frac{1}{3} \text{ AT } (3, \ln 3)$$

$$y = \frac{1}{3}x + \ln 3 - 1$$

DIFFERENTIATION

56.

$$y = \ln(x+2)(x-3)$$

$$y = \ln \frac{x+1}{x+4}$$

$$y = \ln \frac{x^2 - x - 6}{\text{PART}} \quad \text{F.F.}$$

$$y' = \frac{2x-1}{(x^2-x-6)}$$

$$y = \ln(x+1) - \ln(x+4)!$$

LOGARITHM RULE

$$y' = \frac{1}{x+1} - \frac{1}{x+4}$$

TIMES. TIMES. TIMES

$$y' = \frac{3}{(x+1)(x+4)}$$

$$y = \ln x^2$$

$$y = (\ln x)^2$$

MENTALLY

$$y' = \frac{2x}{x^2}$$

WRITE

$$y' = \frac{2}{x}$$

FUNCTION OF A FUNCTION

DER. PART X DER. WHOLE

EYES SEE

BRAIN WORKS

$$y' = \frac{2 \ln x}{x}$$

PEN WRITES.

DIFFERENTIATION

57.

$$y = \ln \sqrt{x^2 + 9}$$

THE PART IS A
FUNCTION OF A
FUNCTION

THE WHOLE IS A
FUNCTION OF A FUNCTION

↓
DERIVATIVE PART X DERIVATIVE WHOLE
DERIVATION

$$y = \ln (x^2 + 9)^{1/2}$$

$$y' = \frac{x(x^2 + 9)^{-1/2}}{(x^2 + 9)^{1/2}} = \frac{x}{x^2 + 9}$$

EXPONENTIAL GROWTH & DECAY 58.

BACTERIA

$$N = N_0 e^{.12t} \text{ IN HOURS}$$

$N_0 = 240000$ N_5 FIND

INITIAL AMOUNT	AMOUNT ↓	AFTER 5 HOURS
-------------------	-------------	------------------

$$N_5 = 240000 e^{.6}$$

$$\doteq 437000$$

(APPR.)

THERE ARE VARIOUS WAYS OF WRITING
THE BASIC FORMULA

$$y = A e^{kt}$$

$$y = y_0 e^{kt}$$

$$N = N_0 e^{kt}$$

$$P = P_0 e^{kt}$$

K IS A CONSTANT (RATE)

$k > 0$ INDICATES GROWTH, $k < 0$ MEANS DECAY

INITIAL VALUES: A, y_0, N_0, P_0 , ETC

EXPONENTIAL GROWTH & DECAY ^{59.}

$$y = Ae^{kt}$$

CALCULATE
RATE OF CHANGE

* A FUNCTION OF TIME

THINK: DIFFERENTIATION
WITH RESPECT TO t

FUNCTION OF A FUNCTION

DER. PART(kt) $\times$ DER. WHOLE

REMEMBER: IF $y = e^t$, $y' = e^t$

A & k ARE CONSTANTS. ONLY t IS THE VARIABLE.

$$y' = kAe^{kt} \quad \text{SUB. } y = Ae^{kt}$$

$$y' = ky$$

RATE OF
CHANGE

EXPONENTIAL GROWTH & DECAY 61.

$$P_0 = 2400$$

ANNUAL GROWTH RATE

$$.02$$

RATE IS PROPORTIONAL
TO POPULATION ($P' = KP$)

$$P_{10} \text{ \& } P_t = 3400$$

$$P_{10} = 2400 e^{.2} = 2931$$

$$3400 = 2400 e^{.02t}$$

$$\text{LOG}_e (34 \div 24) = .02t$$

DEFINITION OF LOGARITHM

$$\text{ROUTINE: } t = \frac{\ln(34 \div 24)}{.02} = 17.4 \text{ YEARS}$$

POPULATION 20904 IN 1971 } FIND
21650 IN 1975 } P_{1987}
RATE AS A PERCENTAGE

$$21650 = 20904 e^{4k}$$

$$4k = \ln(21650 \div 20904) , k = .88\%$$

$$P_{1987} = 20904 e^{.0088 \times 16} \doteq 2400$$

EXPONENTIAL GROWTH & DECAY 61.

VELOCITY 6 m/sec · STOP ENGINE.

DECELERATION PROP. TO V ($V' = kV$)

AFTER 100 SEC., $V = 3$. WHEN WILL V BE 1.

"DECAY"

$$V = V_0 e^{kt} \rightarrow 3 = 6e^{100k}$$

$k \doteq \frac{\text{NEGATIVE}}{.0069314}$

$$1 = 6e^{-.0069314t}$$

$$t \doteq 258 \text{ SECONDS}$$

RADIO ACTIVE MATERIAL. $M_0 = 50 \text{ g}$.

$$M_{10} = 40 \text{ g}$$

FIND k , M_{15} & HALF LIFE

$$40 = 50e^{10k}$$

Y WHEN M IS HALF M_0
FOR ANY R.A.M. $\rightarrow$

$$\ln .5 \div k$$

$$k = \frac{\ln .8}{10}$$

$$= -.0223143$$

LEAVE IN CALC.

TIMES 15

PRESS $e^x \dots \times 50 = \dots$

$$M_{15} = 35.8 \text{ g}$$

$$\text{HALF LIFE: } e^{kt} = .5 \rightarrow t = 31.16 \text{ YEARS}$$

EXPONENTIAL GROWTH & DECAY 62.

\$2000 INVESTED @ 10%
FOR 5 YEARS. INTEREST COMPOUNDED
CONTINUALLY

$$P_5 = 2000 e^{.5} = \$3297.44$$

COMPARE COMPOUNDED ANNUALLY

$$2000 \times 1.1^5 = \$3221$$

$$V_0 = 100 \text{ m/s} \quad . \quad V_4 = 20 \text{ m/s}$$

CALCULATE V_8

DECELERATION

(DECAY)

$$k < 0$$

$$e^{4k} = .2$$

$$k = (\ln .2) \div 4 = \dots \text{ LEAVE,}$$

$$\text{TIMES } 8 = \dots \text{ PRESS } e^x \dots \times 100 = 4$$

$$V_8 = 4 \text{ m/s}$$

EXPONENTIAL GROWTH & DECAY ^{63.}

BACTERIA

$$N_0 = 4.6 \times 10^6$$

$$N_5 = 7.2 \times 10^6$$

CALCULATE

$$K \text{ \& } N_{12}$$

$$e^{5K} = 72 \div 46$$

$$\text{GROWTH RATE } K = \ln(72 \div 46) \div 5 = .189615$$

$$\text{TIMES } 12 = \dots \text{ PRESS } e^x \dots \times 4.6 \times 10^6 \quad \text{LEAVE}$$

$$N_{12} = 1.3 \times 10^7$$

CHEMICAL: DECAY RATE $K = -.0005$

$$M_0 = 10 \text{ g}$$

CALCULATE

HALF LIFE & M AFTER 500 SEC.

$$M_{500} = 10e^{-.25} = 7.79 \text{ g}$$

$$\text{HALF LIFE: } e^{-.0005t} = .5$$

$$t \div 1386 \text{ SECONDS}$$

EXPONENTIAL GROWTH & DECAY 64.

HALF LIFE CARBON 14 = 5730 y

USED FOR DATING OBJECTS < 50 000 y

ANIMALS & VEGETATION, READING 12.5 COUNTS/MIN
HOW OLD COUNT 7.

$$e^{5730k} = .5$$

$$k = \ln .5 \div 5730 = -1.21 \times 10^{-4}$$

$$7 = 12.5 e^{kt}$$

$$t_7 = \frac{\ln(7 \div 12.5)}{k} \div 4792 \text{ y}$$

PLANT, 50 cm, GROWS 10 cm 1st WEEK
THEN 80% OF PREVIOUS GROWTH.

CALCULATE ULTIMATE HEIGHT

G.P. $a = 10$, $r = .8$

$$\text{LIM GROWTH} = \frac{a}{1-r} = 50 \text{ cm}$$

$$\text{ULTIMATE HEIGHT} = 100 \text{ cm}$$

EXPONENTIAL GROWTH & DECAY 65.

BACTERIA $P = 1000e^{.007t}$ (IN MIN)

CALCULATE P_{10} & INSTANTANEOUS RATE OF INCREASE

$$P_{10} = 1073$$

↖ GROWTH
RATE k

$$P'_{10} = 7e^{.07} = 7.5$$

MEANS: WHEN $t=10$, THE COLONY IS INCREASING
AT THE RATE OF 7.5 BACTERIA
PER MINUTE

$$M = Ae^{kt} \quad \text{WHEN } t=0, M=10$$

$$M_5 = 9$$

FIND M_{20}

$$10 = Ae^0 \therefore A = 10$$

$$M_5 = 10e^{5k} = 9$$

$$k = \ln 9 \div 5 = \dots \quad \text{LEAVE IN CALCULATOR}$$

$$\begin{aligned} \text{TIMES } 20 = \dots \text{ PRESS } e^x \dots \times 10 &= 6.561 \\ &= M_{20} \end{aligned}$$

EXPONENTIAL GROWTH & DECAY

66.

ELECTRICAL CIRCUIT

$$I_{\text{AMPS}} = Ae^{-at} - 2\cos\pi t$$

$$I_0 = 0, I_{2.5} = 1.2$$

CALCULATE I_5

$$I_0 = A - 2 = 0 \therefore A = 2$$

$$1.2 = 2e^{-2.5a} - 2\cos\frac{5\pi}{2}$$

$$\cos\frac{5\pi}{2} = 0 \therefore a = \ln.6 \div -2.5 = \dots \text{ LEAVE IN CALC.}$$

$$\text{TIMES } \frac{1}{5} = \dots \text{ PRESS } e^x \dots \times 2 = .72$$

$$\frac{-2\cos 5\pi = 2}{I_5 = 2.72 \text{ AMPS}}$$

$$N_t = Ce^{kt}$$

$$N_3 = 10$$

$$N_{18} = 100$$

$$\text{PROVE } N(at+bu) = (N_t)^a \cdot (N_u)^b$$

FIND N_{13}

WORK BACKWARDS FROM HERE

$$C^a e^{akt} \cdot C^b e^{bku}$$

$$= Ce^{k(at+bu)} = N(at+bu)$$

C^{a+b} IS ALSO A CONSTANT

SAME FORMAT

$$Ce^{3k} = 10 \quad Ce^{18k} = 100$$

$$e^{15k} = 10$$

$$e^k = 10^{1/15}$$

$$e^{13k} = 10^{13/15}$$

$$e^{3k} = 10^{1/5}$$

$$C = 10^{4/5}$$

$$N_{13} = 10^{4/5} e^{13k} = 10^{5/3} = 46.4$$

EXPONENTIAL GROWTH & DECAY

67

1960. POPULATION = 15.1 MIL.
P DOUBLES EV. 20 YEAR.
WHEN 3 P & P₂₀₁₀

$$P_{20} = 15.1 e^{20k} = 30.2$$

$$e^{20k} = 2, \quad k = .034657 \dots$$

RECIPROCAL... $\times \ln 3 = 31.7$
DURING 1991 POP. TREBLES

$$P_{2010} = 15.1 e^{50k}$$

$$\rightarrow k = \ln 2 \div 20 = \dots$$

$$\times 50 = \dots e^x \dots \times 15.1$$

$$P_{\text{IN } 2010} = 85.4 \text{ MIL.}$$

POP. DATA 2 TOWNS (1.1.1980)
 $P_A = 2000 e^{-.02t}$ $P_B = 1000 e^{.03t}$
AN. G.R.B. P_B 1995 INST. RATE. $B \nabla A$ WHEN

ANNUAL GROWTH RATE FOR
 $B = 3\%$, FOR $A = -2\%$

$$P_{B1995} = 1000 e^{.03t}$$

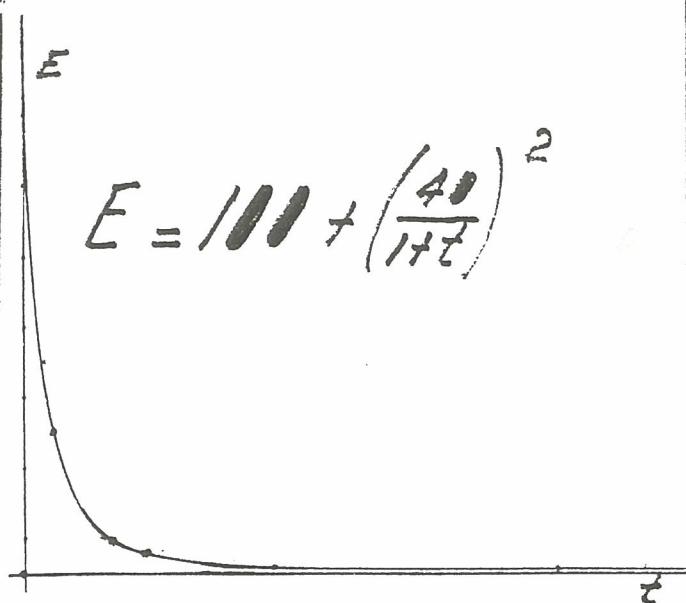
$$P' = 30 e^{.45} \doteq 47$$

MEANS INSTANTANEOUS RATE
OF INCREASE AT 1.1.95 IS
47 PEOPLE P.A.

WHEN WILL $P_B > P_A$.
 $1000 e^{.03t} > 2000 e^{-.02t}$
 $e^{.03t} > 2, \quad t > 713$
DURING 1993 $P_B > P_A$

FROM 1.10.89 EMISSION RATE
IN T.P.Q WILL BE $E = 100 + \left(\frac{40}{1+t}\right)^2$
FIND Lim. E & AMOUNT IN T.

OF CHLORO-FLUORO CARBONS (CFC)
. FIND E 1989 & E 1998
EMITTED DURING THE PERIOD



INSTANTANEOUS RATE
AT 1.1.1989 = 1700 TONNES

AT 1.1.189, 116 TONNES
 $\lim_{t \rightarrow \infty} E = 100$ TONNES P.A.

IF x TONNES IS AVAILABLE
 $\frac{dx}{dt} = E$, MEANS $x = \int_0^9 E dt$

$$= \left[100t - \frac{1600}{1+t} \right]_0^9 = 2340 T$$

THE CHANGE IN THE AMOUNT
IS THE EMISSION

EXPONENTIAL GROWTH & DECAY

68.

RADIUM: $Q = Q_0 e^{-kt}$
 HALF LIFE WHEN $t = 1690$
 FIND k & $\frac{1}{3} Q_0$

TERMITES: $N_t = \frac{A}{2 + e^{-t}}$
 $N_0 = 3 \times 10^5$
 FIND A . t LARGE & RATE

DECAY: $k < 0$
 USE $Q = Q_0 e^{kt}$ (EASIER)
 $e^{1690k} = .5, k = -4.1 \times 10^{-4}$
 LEAVE IN CALCULATOR
 RECIPROCAL $\frac{1}{x} \dots$
 $x / \ln .2 = 3924$ YEARS
 $(e^{kt} = .2)$

$$A = 9 \times 10^5$$

$$\lim_{t \rightarrow \infty} N_t = 4.5 \times 10^5$$

$$N_t = A (2 + e^{-t})^{-1}$$

$$\frac{dN}{dt} = A (2 + e^{-t})^{-2}$$

$$= \frac{9 \times 10^5}{(2 + e^{-t})^2}$$

BOTTLE WITH 200 ml
 20 SEC. TO FILL. FLOWRATE
 $R = 4(20 - t)$. CAPACITY
 BOTTLE & WHEN HALF FULL

RATE FUEL BURN
 $R = 10 + \frac{10}{1 + 2t}$ kg/min
 R_7 , t LARGE & AMOUNT
 BURNT FIRST 7 MIN.

x ml TO FILL

$$R = 80 - 4t$$

$$\frac{dx}{dt} = 80 - 4t$$

$$x = [80t - 2t^2]_0^{20}$$

$$x = 800 \text{ ml, CAP. } 1 \text{ L}$$

$$[80t - 2t^2]_0^t = 300$$

$$t^2 - 40t + 150 = 0 \quad t < 20$$

$$t = 4.2 \text{ SECONDS}$$

$$R_7 = 10 \frac{2}{3} \text{ kg/min.}$$

$$\lim_{t \rightarrow \infty} R_t = 10 \text{ kg/min}$$

x kg BURNT IN 7 MIN

$$R = 10 + 5 \left(\frac{2}{1 + 2t} \right)$$

$$x = [10t + 5 \ln(1 + 2t)]_0^7$$

$$83.54 \text{ kg.}$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

69

$$(a+b)^2$$

(PERFECT) SQUARE

SQUARE THE FIRST
DOUBLE THE PRODUCT
SQUARE THE LAST

$$a^2 + 2ab + b^2$$

$$(a-b)^2$$

ROUTINE

$$a^2 - 2ab + b^2$$

$$(a+5)^2$$

$$a^2 + 10a + 25$$

$$(a-4)^2$$

$$a^2 - 8a + 16$$

$$x^2 - 16$$

THE DIFFERENCE OF
2 SQUARES

$$(x-4)(x+4)$$

$$3x^2 - 12$$

$$3(x-2)(x+2)$$

$$(2-\sqrt{3})(2+\sqrt{3})$$

THE REVERSE

$$2^2 - (\sqrt{3})^2$$

/

$$(x+1)^2 - (x-1)^2$$

$$4x$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

70.

$$a^3 - b^3$$

$$(a-b)(a^2+ab+b^2)$$

SINGLE
PRODUCT

$$a^3 + b^3$$

SAME → OPPOSITE

$$(a+b)(a^2-ab+b^2)$$

$$(x-2)(x+5)$$

$$x^2 - x - 12$$

TYPE
1

$$(x-2)(x+5)$$

COMPARE
24
X 35

TRINOMIAL
LOOK FOR & FIND 2 NUMBERS:
PRODUCT - 12, SUM - 1

COMBINE 2 & 3 MENTALLY!

$$x^2 + 3x - 10$$

$$(x-4)(x+3)$$

$$2x^2 - 5x - 12$$

TYPE
2

$$6x^2 + 10x - 4$$

TYPE
3

PRIME

THE CROSS

WITH FORESIGHT

$$\begin{array}{cc} (2x & +3) \\ & \diagdown \quad \diagup \\ (x & -4) \end{array}$$

COMPOSITE

CROSS PRODUCTS
-2X + 12X = 10X

END PRODUCT

END PRODUCT

$$\begin{array}{cc} (3x & -1) \\ & \diagdown \quad \diagup \\ (2x & +4) \end{array}$$

$$x(x-2) + 3(x-2)$$

$$ab^2 + ac - b^2d - cd$$

$$(x+3)(x-2)$$

$$a(b^2+c) - d(b^2+c)$$

COMPARE:

$$\begin{array}{l} 89 \times 13 + 11 \times 13 \\ (89+11)(13) = 1300 \end{array}$$

ADJUST IF NECESSARY

$$(a-d)(b^2+c)$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

71.

$$\frac{x^2 - 5x}{x^2 - 25}$$

$$\frac{x(x-2) + 3(x-2)}{x^2 - 4}$$

$$\frac{x}{x+5}$$

$$x \neq 5$$

$$\frac{x+3}{x+2}$$

$$x \neq 2$$

$$\frac{x^2 + 3x + 2}{x^2 + 5x + 6}$$

$$x^4 - 13x^2 + 36 = 0$$

$$\frac{(x+2)(x+1)}{(x+2)(x+3)}$$

$$\frac{x+1}{x+3}$$

$$x \neq -2$$

$$(x^2 - 9)(x^2 - 4) = 0$$

$$x = \pm 3, \quad x = \pm 2$$



$$x(2x-3) = 5$$

$$3x^2 = 5x - 2$$

$$2x^2 - 3x - 5 = 0$$

THE CROSS: $(2x-5)(x+1) = 0$

$$x = -1, \quad x = 2\frac{1}{2}$$

THE FORMULA: SUB. MENTALLY

$$x = \frac{3 \pm \sqrt{9+40}}{4} \quad \text{SAME (DIRECT) ANSWERS}$$

PARABOLA
STRAIGHT LINE
INTERSECTION (GRAPH)

ALL PURPOSE FORMULA

$$3x^2 - 5x + 2 = 0 \quad x = 1, \quad x = \frac{2}{3}$$

$$x + \frac{1}{x} - 3 = 0$$

ROOTS IN
SURDS

$$px^2 - x + q = 0$$

ROOTS
-1 & 3

$$x^2 - 3x + 1 = 0$$

↓ THINK
FORMULA

$$x = \frac{3 \pm \sqrt{5}}{2}$$

$$\alpha + \beta = \frac{1}{p} = 2 \quad (1)$$

$$\alpha \beta = \frac{q}{p} = -3 \quad (2)$$

$$p = \frac{1}{2}, \text{ SUB (2), } q = -\frac{3}{2}$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

72.

$$-x > -2$$

↓ THINK FLIP

THE ROUTINE

$$x < 2$$

THE UNDERSTANDING

$$-4 > -12, \text{ BUT } 4 < 12$$

$$3 - 2x \leq 7$$

x NEG, ↓ THINK FLIP

$$x \geq -2$$



$$\frac{-x}{4} + 2 \leq 1$$

$$\frac{1}{x} > \frac{1}{5}$$

MENTALLY!

$$\frac{-x}{4} \leq -1$$

WRITING DOWN PURE ARITHMETIC IS TIME CONSUMING & UNNECESSARY!

WRITE

$$x \geq 4$$

x IN DENOMINATOR:

THINK FLIP AND $x \neq 0$

AND $x < 5$ AND $x > 0$

$$0 < x < 5 \quad \text{TEST } x=3$$

$$\frac{6}{x} < 3$$

$$x \neq 0$$

$$\frac{x+3}{x} < 4$$

$$\frac{6}{x} < \frac{6}{2}$$

$$\frac{1}{x} < \frac{1}{2}$$

$$x > 2 \text{ \& } x < 0$$



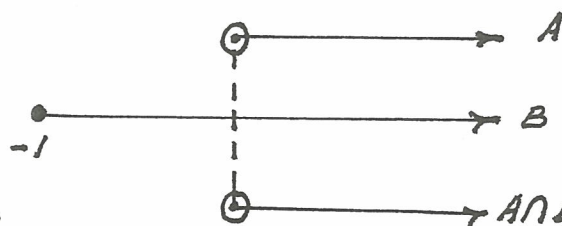
$$\begin{cases} 1 + \frac{3}{x} < 4 \\ \frac{3}{x} < 3 \end{cases} \begin{cases} x > 1 \\ x < 0 \end{cases} \quad x \neq 0$$

$$\lim_{x \rightarrow \infty} \frac{x+3}{x} = 1 + \frac{3}{x} = 1$$

$$\{x: x > 3\}^A \cap \{x: x > -1\}^B$$

$$\frac{2x-1}{3} < \frac{3x-4}{2} < 5 \quad x \in \mathbb{N}$$

INTERSECTION



$$\{x: x > 3, x \text{ REAL}\}$$

TIMES. TIMES. TIMES

LEFT. RIGHT. RIGHT

$$4x - 2 + 9x - 12 < 30$$

$$x < 3\frac{5}{13} \quad \text{TRUTH SET } \{0, 1, 2, 3\}$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

73.

$$-x^2 - 3x + 4 < 0$$

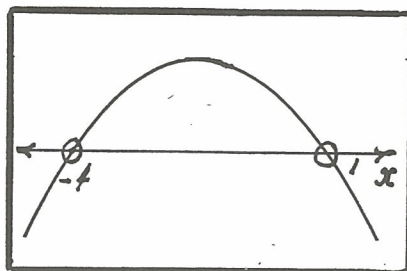
FOR WHAT VALUES OF x IS THE PARAB
 $y = -x^2 - 3x + 4$ BELOW THE x -AXIS

$$(x+4)(x-1) > 0$$

TEST $x=0$, FALSE
 ASSUME TO FIND



CONCAVE DOWN



$$x < -4$$

$$x > 1$$

$$\frac{xy^{-1} + yx^{-1}}{x^2 + y^2}$$

$$\frac{(8x)^{-1}}{2^{-6}}$$

$$\frac{\frac{x}{y} + \frac{y}{x}}{x^2 + y^2}$$

$$\frac{1}{8x} \times 64 = \frac{8}{x}$$

$$\frac{1}{xy}$$

$$x^2 + y^2 \neq 0$$

$$mx^2 - 20x + m = 0$$

$$x^2 - 4x > 0$$

ONE ROOT IS 3

$$\alpha + \beta = \frac{-b}{a} = \frac{20}{m}$$

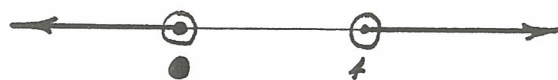
$$\alpha\beta = \frac{c}{a} = 1$$

ASSUME EQUAL TO ESTABLISH
 BOUNDARIES (NOT INCLUDED)

$$x(x-4) = 0$$

ASSUME $\beta = 3$, $3\alpha = 1$, $\alpha = \frac{1}{3}$
 $\therefore m = 6$

TEST 2, FALSE



$$\log_2 8$$

$$(8x^6)^{1/3} \cdot x^{-3}$$

CALL IT x AND
 "GO AROUND THE BLOCK"

TO UNDERSTAND

$\log_2 8 = x$ MEANS $2^x = 8$
 $= 2^3 \therefore x = 3$

$$2x^2 \times x^{-3} = \frac{2}{x}$$

REMEMBER: $8^{1/3} = \sqrt[3]{8} = 2$

$$\ln 8 \div \ln 2 = 3 \quad \ln 2 \div \ln 2 = 1$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

74.

$$\sqrt{7} \times \sqrt{63}$$

$$\sqrt{7} \times 3\sqrt{7} = 21$$

SURDS: LOOK FOR
SQUARE FACTORS
TO SIMPLIFY

$$\frac{6 + \sqrt{2}}{3 + \sqrt{2}}$$

THINK DIFFERENCE 2 SQUARES

$$\therefore \text{TIMES } \frac{3 - \sqrt{2}}{3 - \sqrt{2}}$$

$$\frac{16 - 3\sqrt{2}}{7}$$

$$\sqrt{45} + \sqrt{80} = a\sqrt{5}$$

$$\frac{4 + \sqrt{3}}{2 + \sqrt{3}} = b - 2\sqrt{3}$$

$$7\sqrt{5} = a\sqrt{5}$$

$$a = 7$$

$$(4 + \sqrt{3})(2 - \sqrt{3}) = b - 2\sqrt{3}$$

$$5 - 2\sqrt{3} = b - 2\sqrt{3}$$

$$b = 5$$

$$\frac{4}{2 + \sqrt{5}} - \frac{1}{9 - 4\sqrt{5}}$$

$$8^{-2/3} \times 4^{3/2}$$

$$\frac{4(9 - 4\sqrt{5}) - (2 + \sqrt{5})}{(2 + \sqrt{5})(9 - 4\sqrt{5})}$$

$$\frac{-17(\sqrt{5} - 2)}{(\sqrt{5} - 2)} = -17$$

$$2^{-2} \times 2^3 = 2$$

$2^{-\frac{1}{2}k}$	$2^{\frac{1}{2}k}$	2^k	2^{-k}	$k > 0$
---------------------	--------------------	-------	----------	---------

$$y = 2x^2 + 3x + 5$$

ASSUME $k = 4$

OR
SQUARE

↑ $A > 0$ ARMS UP

$\frac{1}{4}$	4	16	$\frac{1}{16}$
---------------	---	----	----------------

$$\Delta = 9 - 40$$

NO REAL ROOTS

2^k LARGEST. 2^{-k} SMALLEST

PARABOLA: POSITIVE DEFINITE

EXPRESSIONS · EQUATIONS · INEQUALITIES

75.

$$\frac{A^4 C}{B^4} \quad A = \left(\frac{2}{3}\right)^2 \quad B = \left(\frac{4}{3}\right)^4 \quad C = \left(\frac{8}{3}\right)^7$$

$$25^{-\frac{1}{2}}$$

$$\log_4 16$$

$$-2 - -3$$

EXPRESS AS POWERS OF 2 & 3
USE INDEX RULES TO SIMPLIFY

$$\frac{2^8}{3^8} \times \frac{2^{21}}{3^7} \times \frac{3^{16}}{2^{32}} = \frac{3}{8}$$

$$\frac{1}{5}$$

$$\log_4 16 = x$$

MEANS

$$4^x = 16$$

$$x = 2$$

$$\ln 16 \div \ln 4 = 2$$

$$-2 + 3$$

$$1$$

$$\frac{\sqrt{32} - \sqrt{8}}{3\sqrt{2}}$$

$$\frac{2}{15} - \frac{275}{1000}$$

$$\frac{1}{2}(x+2) - \frac{1}{5}(x-3) = 1$$

$$\frac{2\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$$

$$\frac{2}{15} - \frac{11}{40}$$

$$5x + 10 - 2x + 6 = 10$$

$$x = -2$$

$$- \frac{17}{120}$$

$$-3 < 1 - x < 4$$

$$\frac{x+1}{x^2-x}$$

$$\frac{x-1}{x^2+x}$$

INTERSECTION OF
2 INEQUALITIES

$$1 - x > -3$$

$$x < 4$$

$$1 - x < 4$$

$$x > -3$$

$$-3 < x < 4$$

$$\frac{(x+1)(x^2+x) - (x^2-x)(x-1)}{x^2(x-1)(x+1)}$$

$$\frac{4}{(x-1)(x+1)}$$

$$T = 78125$$

$$357.896$$

$$\left(36^{\frac{1}{4}}\right)^2$$

$$3^{3.5}$$

$$15\% \text{ OFF}$$

$$\text{OR } \$20$$

$$20\% \text{ INCR.}$$

$$\text{NOW } \$90$$

$$78125$$

$$x^4 \text{ (CASIO)}$$

$$\frac{1}{7}$$

$$5$$

$$3.6 \times 10^2$$

$$357.90$$

2 D.P

$$358$$

NEAREST WHOLE

$$36^{\frac{1}{2}}$$

$$\sqrt{36}$$

$$6$$

$$3x^4$$

$$3.5$$

$$46.77$$

$$\frac{15}{100} = \frac{20}{x}$$

PAY

$$\$133.35$$

↑

$$\frac{12}{10} = \frac{90}{x}$$

WAS

$$\$75$$

EXPRESSIONS · EQUATIONS · INEQUALITIES

76.

$$4x^4 = 4x^2 + 3$$

$$\frac{1}{5}(t^3 - t^2 + 1) \quad t = -3$$

$$(2x^2 - 3)(2x^2 + 1) = 0$$

$$x = \pm \sqrt{\frac{3}{2}} \quad x^2 = -\frac{1}{2}$$

N.A.

$$\frac{1}{5}(-27 - 9 + 1)$$

-7

$$\frac{x^{-1} + y^{-1}}{x + y}$$

$$x - \frac{1}{x} \quad \text{WHEN } x = \sqrt{2} + 1$$

$$\frac{\frac{1}{x} + \frac{1}{y}}{x + y} = \frac{\frac{x + y}{xy}}{x + y}$$

$$\frac{1}{xy} \quad x + y \neq 0$$

$$\frac{x^2 - 1}{x} = \frac{(x - 1)(x + 1)}{x}$$

$$\frac{\sqrt{2}(\sqrt{2} + 2)}{\sqrt{2} + 1} = 2$$

$$\frac{3x - 1}{5x + 1} = \frac{3x - 2}{5x + 2}$$

$$x^2 + (k - 3)x + k > 0$$

CROSS MULTIPLY

$$15x^2 + x - 2 = 15x^2 - 7x - 2$$

$$x = 0$$

FOR ALL x
 A) 0 PARABOLA
 CONCAVE UP
 POSITIVE DEFINITE
 $\Delta < 0 \quad k^2 - 10k + 9 < 0$
 $1 < k < 9$ TESTPOINT

$$x^2 - (k + 3)x + (k + 6) = 0$$

REAL ROOTS

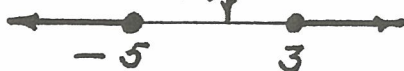
EQUAL
DIFFERENT

$$k^2 + 6k + 9 - 4k - 24 > 0$$

$$(k + 5)(k - 3) > 0$$

TEST $k = 0$

A) 0. POS. DEF.
NO REAL ROOTS



$$\Delta > 0$$

Δ CAPITAL D
GREEK

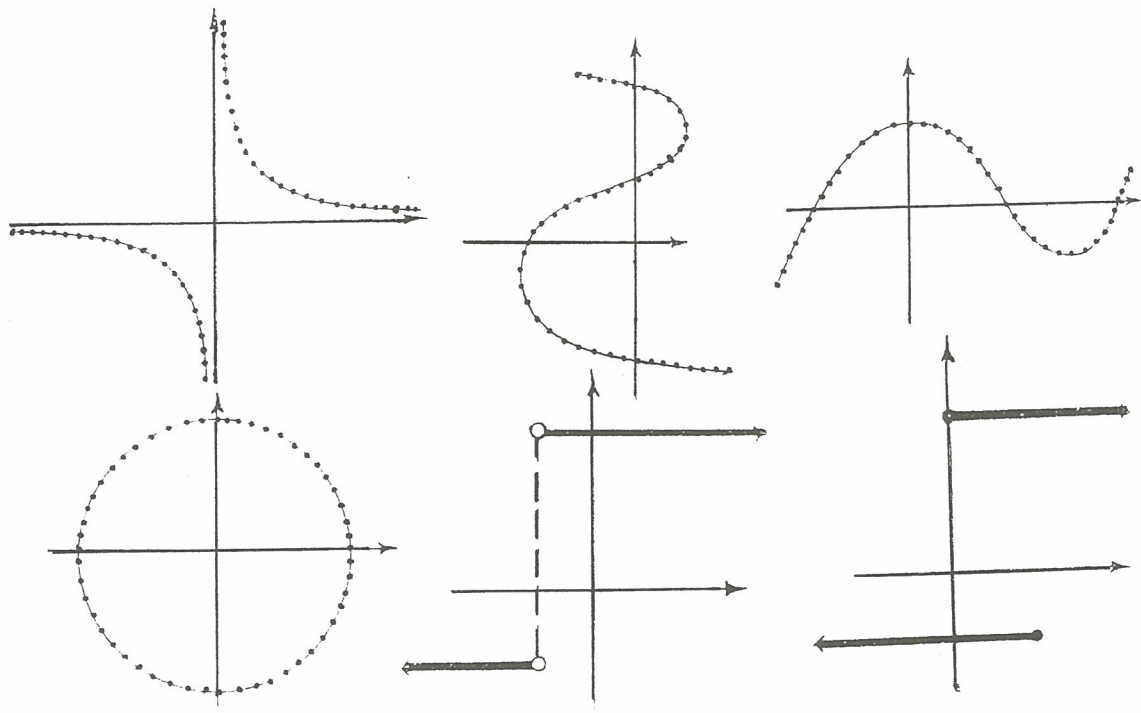
THINK
DISCRIMINANT
 $\Delta = b^2 - 4ac$

IT DISCRIMINATES
 NO ROOTS
 EQUAL - OR
 DIFFERENT ROOTS

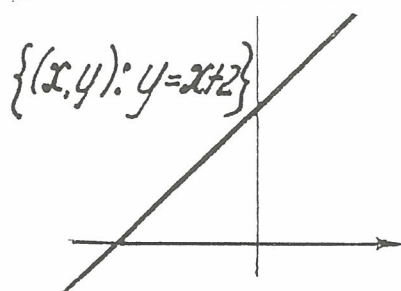
FUNCTIONS & RELATIONS 77.

ALL FUNCTIONS ARE RELATIONS, BUT NOT
NOT V.V.

VERTICAL LINE TEST: ONE x
ONE y

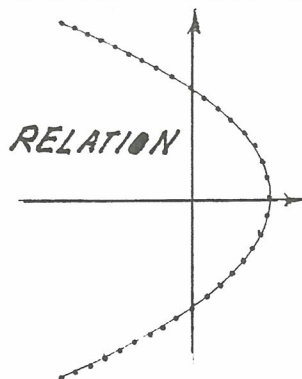


RELATION &
FUNCTIONS WRITTEN IN SET NOTATION

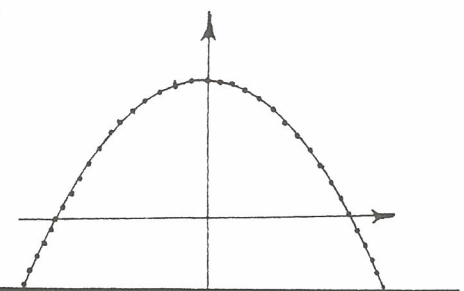


THE SET OF x 's & y 's
SUCH THAT

$$y = x + 2$$



$\{(x, y): x = -y^2 + 4\}$
NEG. OPENING
LEFT



$\{(x, y): y = -x^2 + 4\}$
PARABOLA
A NEG. ARMS DOWN

FUNCTIONS & RELATIONS

78.

$y = x^2$ EVEN	$y = x^3$ ODD	$y = x^3 + 2$ NEITHER	$y = x - 1$ INVERSE
TEST $x = 1$ $y = 1$ TEST $x = -1$ $y = 1$	TEST $x = 1$ $y = 1$ TEST $x = -1$ $y = -1$	TEST $x = 1$ $y = 3$ TEST $x = -1$ $y = 1$	INTERCHANGE x & y $x = y - 1$ OR $y = x + 1$ SYMMETRICAL ABOUT $y = x$
$x \in \{0, 1, 2, 3\}$ $f(x) = x - 4$		FIND $h(3)$ IF $h(x) = (x - 1)(x - 4)$	
$f(0) = -4$ $f(1) = -3$ $f(2) = -2$		$h(3) = -2$ THE FUNCTION OF 3 = -2	
DOMAIN & RANGE x VALUES $f(x) = y$ VALUES		FIND $f(x)$ IF $f(x + 1) = x - 4$	
$y = \sqrt{x + 2}$	$x \geq -2, y \geq 0$	$(x + 1)$ HAS TO BE REPLACED BY x , HENCE REWRITE. $f(x + 1) = (x + 1) - 5$ $f(x) = x - 5$	
$y = -2x^2$	ALL $x, y \leq 0$		
$f(x) = x + x $	$f(x) \geq 0$		
$f(x) = \sqrt{16 - x^2}$	$0 \leq f(x) \leq 4$		

FUNCTIONS & RELATIONS

79

$$f(x) = 2x^2 - 3x$$

$$f(x) = x^2$$

$$\frac{f(x+h) - f(x)}{h}$$

$$\frac{(2(x+h)^2 - 3(x+h) - 2x^2 + 3x) \div h}{4x - 3 + 2h}$$

$$\frac{f(x) - f(3)}{x - 3}$$

$$\frac{x^2 - 9}{x - 3} = x + 3$$

$$x \neq 3$$

$$f(x) = ax^2 + bx + c$$

$$f(x) = \begin{cases} 0 & \text{FOR } x \leq -2 \\ -1 & \text{FOR } -2 < x < 0 \\ x & \text{FOR } x \geq 0 \end{cases}$$

$$f(x) - f(-x)$$

$$ax^2 + bx + c - a(-x)^2 + b(-x) - c = 2bx$$

PIECEMEAL FUNCTION

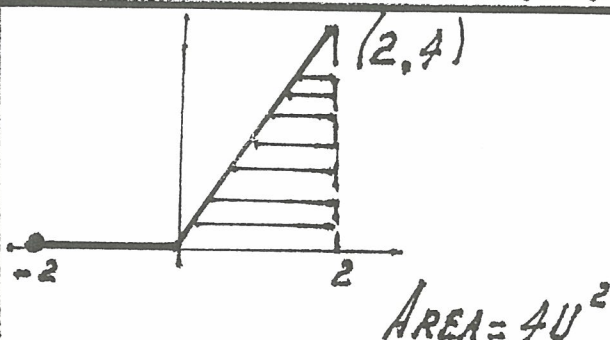
$$f(-2) + f(-1) + f(0) = -1$$

$$f(a^2) \geq 0 \therefore f(a^2) = a^2$$

$$f(x) = \begin{cases} 0 & \text{IF } x \leq 0 \\ 2x & \text{IF } x > 0 \\ & -2 \leq x \leq 2 \end{cases}$$

DOMAIN ALL REAL x

$$f(x) = \frac{1}{1+x^2} \quad \begin{matrix} \text{FIND} \\ \text{RANGE} \end{matrix}$$



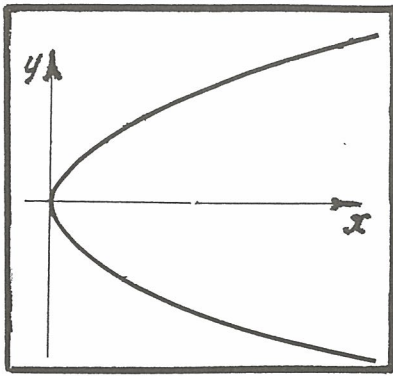
$$x^2 \geq 0, \quad f(0) = 1$$

$$\lim_{x \rightarrow \infty} f(x) = 0$$

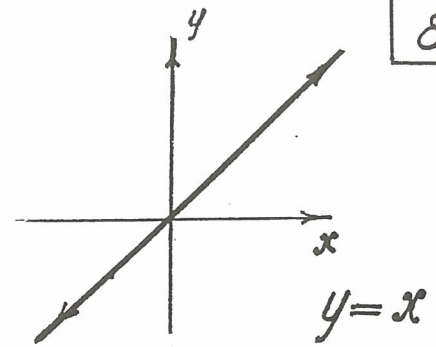
$$x \rightarrow \infty$$

RANGE

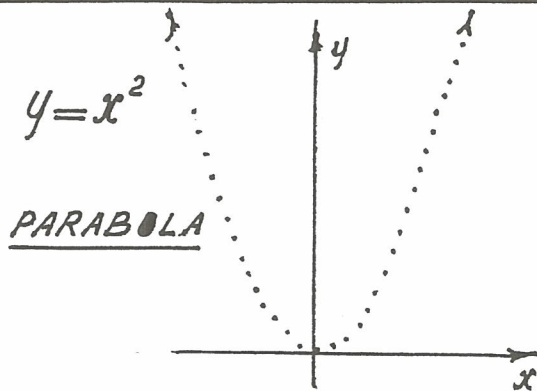
$$0 < y \leq 1$$



$$y^2 = 4x$$



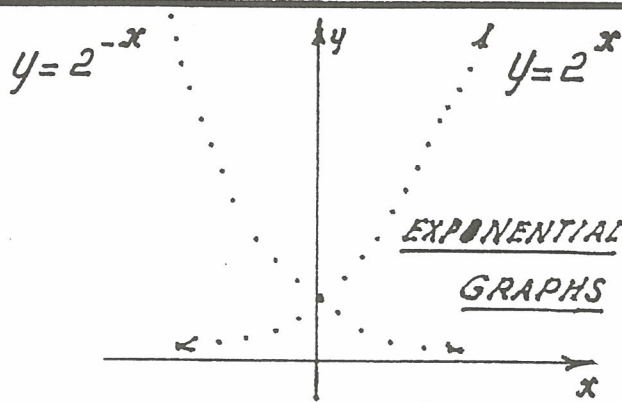
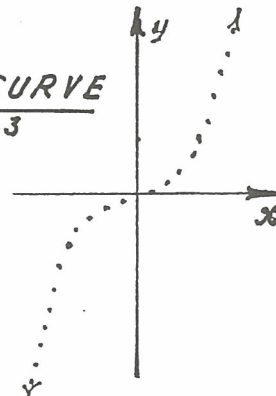
80.



PARABOLA

CUBIC CURVE

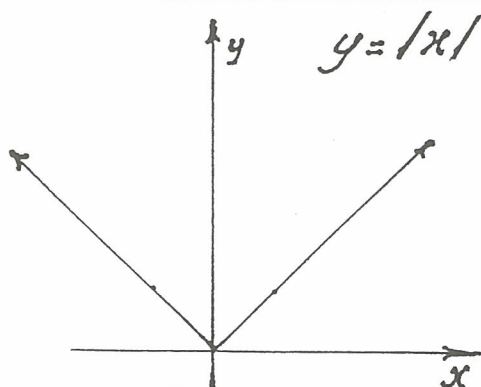
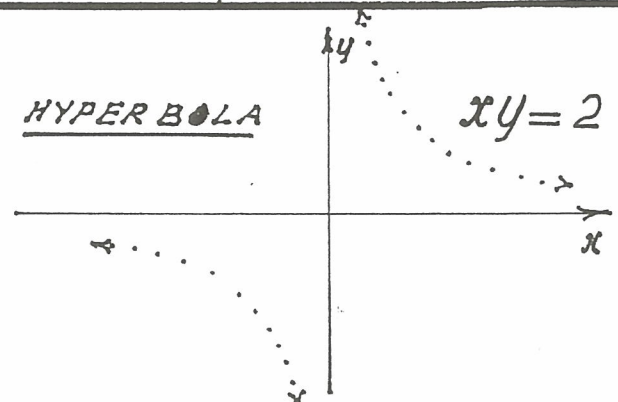
$$y = x^3$$



EXPONENTIAL
GRAPHS

HYPERBOLA

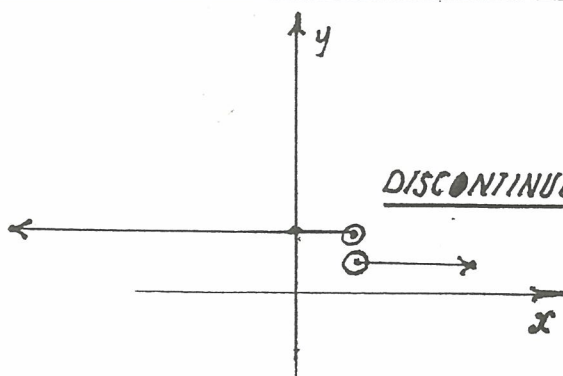
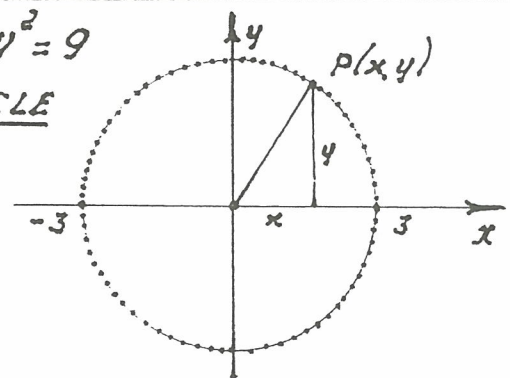
$$xy = 2$$



$$y = |x|$$

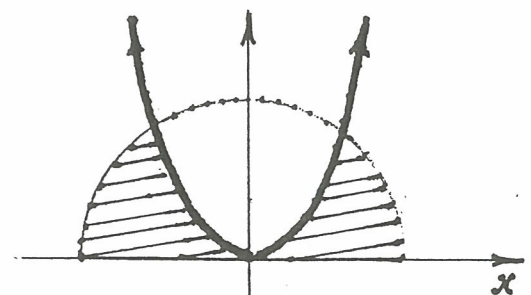
$$x^2 + y^2 = 9$$

CIRCLE



DISCONTINUOUS

PIECE MEAL GRAPH



INTEGRATION

81.

IS USED

- TO FIND AREAS UNDER CURVES
- TO FIND VOLUME OF ROTATED AREAS ABOUT x -OR y -AXES

INTEGRATION INVOLVES ANTI ^{DERIVATION} DIFFERENTIATION

↑ NO R! COMPARE INTEGRITY (WHOLE NESS)
& INTEGERS (WHOLE NUMBERS)
SOME COUNTRIES INTEGRATE OTHERS
MEANS: THEY ANNEXE THEIR AREAS

PRIMITIVES OR INDEFINITE INTEGRALS

$$y = x^2 + 4x + 10$$

$$y = x^2 + 4x$$

$$y = x^2 + 4x - 6$$

THEY ALL HAVE
THE SAME
DERIVATIVE $2x + 4$

IF WE GO BACK FROM y' TO y .

WE WRITE $F(x) = x^2 + 4x + C$ FOR
CONSTANT
BECAUSE THE EXACT
NUMBER IS UNKNOWN.

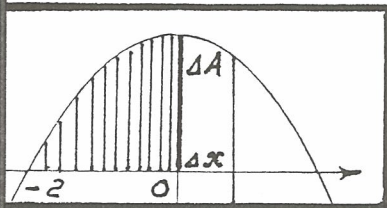
INTEGRATION

82.



THE AREA UNDER THE CURVE
IS THE SUM OF THE AREAS
OF AN INFINITE NUMBER OF RECTANGLES

SYMBOL $\int$



(ENLARGED)
 ΔA IS A SMALL INCREMENT.
AS ITS BASE Δx

APPROACHES 0, $\lim_{\Delta x \rightarrow 0} \frac{\Delta A(x)}{\Delta x} = \frac{dA(x)}{dx} = f(x)$

MEANS THE ORDINATE WHERE $x=0$

THE LIMIT OF THE AREA IS A LINE

CONCLUSION IF THE DERIVATIVE
OF THE AREA $= f(x)$,

THE PRIMITIVE (ANTI DERIVATIVE)
OF $f(x)$ IS THE AREA!

INTEGRATION

83.

THE ROUTINE

ADD 1 TO THE INDEX
AND DIVIDE BY THE NEW ONE

$$y' = x^6$$

$$f(x) = -4x^{-5}$$

$$y = \frac{x^7}{7} + C$$

$$F(x) = x^{-4} + C$$

$$y' = 4x^3 + 3x^2 - 2x + 6$$

$$f(x) = 2x^{-\frac{1}{2}}$$

$$y = x^4 + x^3 - x^2 + 6x + C$$

$$F(x) = 4x^{\frac{1}{2}} + C$$

INTEGRATION

84.

$$\int dx$$

$$\int \sqrt{x}(7x^2+5x-3)dx$$

MEANS $\int 1 dx$

THE PRIMITIVE OR
THE INDEFINITE INTEGRAL
OF 1 IS $x+C$

$$\int (7x^{\frac{5}{2}} + 5x^{\frac{3}{2}} - 3x^{\frac{1}{2}}) dx$$

$$2x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - 2x^{\frac{3}{2}}$$

$$2x^{\frac{3}{2}}(x^2+x-1) + C$$

$$\int (3x-2)(x+3) dx$$

$$\int \frac{x^3-2}{x^2} dx$$

$$\int (3x^2+7x-6) dx$$

$$x^3 + \frac{7x^2}{2} - 6x + C$$

$$\int (x - 2x^{-2}) dx$$

$$\frac{1}{2}x^2 + \frac{2}{x} + C$$

$$\int \frac{\sqrt{x}}{x^4} dx$$

INDEFINITE INTEGRAL
OF $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$$\int x^{-\frac{7}{2}} dx$$

USE $+\frac{2}{2}$

$$-\frac{2}{5}x^{-\frac{5}{2}} + C$$

CHECK
RECIPROCAL

$$\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

INTEGRATION

85

GRADIENT FUNCTION
OF A CURVE THROUGH (1,8) IS
 $2x+5$

SECOND DERIVATIVE
 $\frac{d^2y}{dx^2} = 3x$ AT (1,6)
 $\frac{dy}{dx} = 4$

$y = x^2 + 5x + C$
SUB (1,8), $C = 2$
EQUATION: $y = x^2 + 5x + 2$

$y' = \frac{3}{2}x^2 + C$ SUB $y' = 4$
AT $x = 1$
 $\therefore C = \frac{5}{2}$
 $y = \frac{1}{2}x^3 + \frac{5}{2}x + C$ SUB (1,6)
 $y = \frac{1}{2}x^3 + \frac{5}{2}x + 3$

INTEGRATION

FUNCTION OF A FUNCTION

ROUTINE

SAY & REMEMBER

PRIMITIVE WHOLE

DERIVATIVE PART

INDEFINITE INTEGRAL
OF $(2x+3)^4$

$\int e^{5x+1} dx$

$\frac{(2x+3)^5}{5} + C$
BINGO!

$\frac{e^{5x+1}}{5} + C$
BINGO!

INTEGRATION

86.

TRIGONOMETRIC DERIVATIVES & ANTI DERIVATIVES TO REMEMBER

SIN	COS	-COS	TAN
↕	↕	↕	↕
POS COS	-SIN	SIN OR	SEC ²

PRIMITIVE OF
 $\text{COS } 3x + \text{SIN } 2x$

INDEFINITE INTEGRAL
OF
 $\text{SEC}^2(4x+3)$

FUNCTION OF A FUNCTION
PRIMITIVE WHOLE
DERIVATIVE PART

FUNCTION OF A FUNCTION
PRIMITIVE WHOLE
DERIVATIVE PART

$$\frac{1}{3} \text{SIN } 3x \quad \frac{1}{2} \text{COS } 2x$$

↑

$$\frac{1}{4} \text{TAN}(4x+3)$$

INTEGRATION

87

$$\int \frac{1}{x} dx$$

$$\int \frac{1}{2x} dx$$

THE NORMAL ROUTINE
DOES NOT WORK HERE
DIVISION BY 0 IS OUT!

RECOGNISE AND
WRITE $\ln x + C$

$$\frac{1}{2} \ln x + C$$

$$\int \frac{2x}{x^2+1} dx$$

$$\int \frac{6x}{x^2+7} dx$$

1. THINK LOG
2. IT IS ALREADY IN THE
FORM $\frac{y'}{y}$

ADJUST TO $3 \times \frac{2x}{x^2+7}$
MENTALLY.
OF COURSE!

3. WRITE $\ln(x^2+1) + C$

$$3 \ln(x^2+7) + C$$

$$\int \frac{dx}{x-7}$$

$$\int \frac{x}{1+x^2} dx$$

MEANS: $\int \frac{1}{x-7} dx$
ALREADY IN THE FORM $\frac{f'x}{fx}$

ADJUST MENTALLY TO
 $\frac{1}{2} \times \frac{2x}{1+x^2} = \frac{1}{2} \frac{f'x}{fx}$

$$\ln(x-7) + C$$

$$\frac{1}{2} \ln(1+x^2) + C$$

INTEGRATION

88.

$$\int (x^2 + \frac{2}{x}) dx$$

THINK LOG

$$\frac{x^3}{3} + 2 \ln x + C$$

$$\int \frac{2 \cos 2x}{1 + \sin 2x}$$

ALREADY IN THE FORM

$$\frac{f'(x)}{f(x)}$$

$$\ln(1 + \sin 2x) + C$$

INTEGRAND

$$(x-1)/(x-2)$$

INTEGRAND

$$3 + 2x - x^2$$

$$x^2 - 3x + 2$$

$$\frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$$

$$3x + x^2 - \frac{x^3}{3} + C$$

PRIMITIVE OF

$$\frac{1}{6x-7} - \frac{5}{x}$$

PRIMITIVE OF

$$\frac{5x^2 + 7x - 6}{x^2} dx$$

$$\frac{1}{6} \ln(6x-7) - 5 \ln x + C$$

$$5 + \frac{7}{x} - 6x^{-2}$$

$$5x + 7 \ln x + \frac{6}{x} + C$$

INTEGRATION

89.

THE PRIMITIVE OF

$$4x+3$$

A REAL DETECTIVE SUM

THE ANSWER TO A PROBLEM IS IN
THE PROBLEM, SO KEEP LOOKING AT IT.

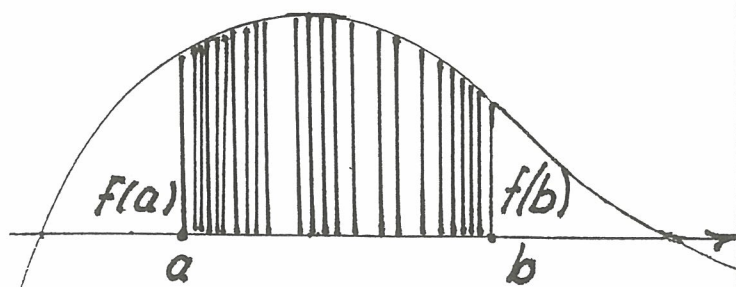
$$2x+1$$

THEN
REWRITE AS $\frac{4x+2}{2x+1} + \frac{1}{2x+1}$

SIMPLIFY &
MODIFY

$$2 + \frac{1}{2} \frac{2}{2x+1}$$

$$2x + \frac{1}{2} \ln(2x+1) + C$$



**DEFINITE
INTEGRAL**
DEFINITE AREA

FROM PAGE
82

$$A(x) = F(x) + C$$

WHEN $x=a$, $A=0 \therefore C = -F(a) \therefore$

$$A(b) = F(b) - F(a)$$

AREA between $f(b)$ & $f(a)$ $= \int_a^b f(x) dx = F(b) - F(a)$

b UPPER LIMIT

INTEGRATION

90.

$$\int_1^3 2x^3 dx$$

CONSIDER

HEIGHT RECTANGLES
AT $x=3\frac{1}{2}$

$f(x)$
BASE
 Δx

$$\left[\frac{x^4}{2} \right]_1^3 = \left(\frac{81}{2} \right) - \left(\frac{1}{2} \right) = 40$$

$$\int_1^K 2x dx = 15$$

$$\left[x^2 \right]_1^K = 15$$

$$K^2 - 1 = 15$$

$$K = 4$$

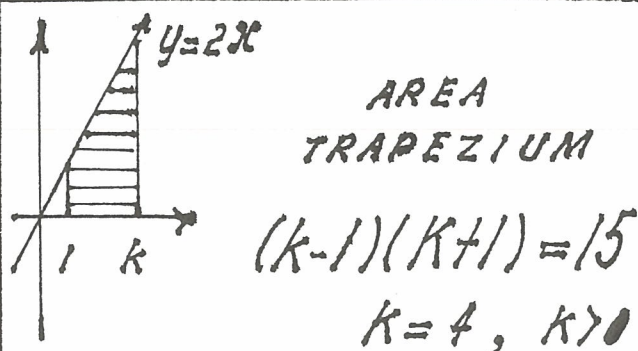
$$\int_1^4 \left(x + \frac{1}{x^2} \right) dx$$

CHECK
WITH COORDINATE GEOMETRY

$$\int_1^4 (x + x^{-2}) dx$$

$$\left[\frac{x^2}{2} - \frac{1}{x} \right]_1^4$$

$$\left(7\frac{3}{4} \right) - \left(-\frac{2}{4} \right) = 8\frac{1}{4}$$

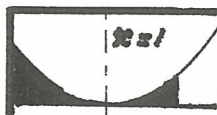


$$\int_{-1}^2 (x-1)^2 dx$$

$y = \frac{1}{2} e^{x^2}$. FIND y' AND
HENCE $\int_{-1}^2 x \cdot e^{x^2} dx$

$$\left[\frac{x^3}{3} - x^2 + x \right]_{-1}^2$$

$$() - () = 3$$



$y' = x \cdot e^{x^2}$
BACK TO y !

$$\left[\frac{e^{x^2}}{2} \right]_{-1}^2$$

$$\frac{e^4 - e}{2}$$

INTEGRATION

91.

$$\int_1^9 (x^{\frac{1}{5}} - x^{\frac{1}{3}}) dx$$

USE $\frac{1}{5} \neq \frac{1}{3}$

$$\int_{\pi/2}^{\pi} (\sin x) dx$$

$$\left[\frac{5}{6} x^{\frac{6}{5}} - \frac{3}{4} x^{\frac{4}{3}} \right]_1^9$$

CHECK RECIPROCALLS
 $\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$

$$\left[-\cos x \right]_{\pi/2}^{\pi}$$

$1 u^2$

$$\int_1^3 \frac{x}{1+x^2} dx$$

$$\int_1^2 \frac{x^2 - 2x + 1}{x} dx$$

$$\frac{1}{2} \left[\ln(1+x^2) \right]_1^3$$

$\uparrow \frac{1}{2} (\ln 10 - \ln 1) = \frac{1}{2} \ln 10$

$$\left[\frac{x^2}{2} - 2x + \ln x \right]_1^2$$

$\ln 2 - \frac{1}{2}$

$$\int_9^{13} \frac{dx}{x-7}$$

$$\int_0^{\pi} (2\sin x - \sin 2x) dx$$

MEANS $\frac{1}{x-7} dx \left(\frac{f'x}{fx} \right)$

$$\left[-2\cos x + \frac{1}{2}\cos 2x \right]_0^{\pi}$$

$$\left[\ln(x-7) \right]_9^{13} = \ln 6 - \ln 2 = \ln 3$$

$$(2 + \frac{1}{2}) - (-2 + \frac{1}{2}) = 4 u^2$$

INTEGRATION

92.

$$\int_2^7 \frac{4}{x+3} dx$$

$$y' = 6t^2 - 2$$

$y=1$ WHEN $t=1$

$$4 \left[\ln(x+3) \right]_2^7 = 4 \ln 2 \text{ u}^2$$

$4 \frac{f'x}{fx}$

$$y = 2t^3 - 2t + C = 1$$

$t=1 \therefore C=1$

$$y = 2t^3 - 2t + 1$$

WHEN $t=2, y=13$

$$\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$\int_{\pi/3}^{\pi} \cos \frac{1}{2}x dx$$

COMPOSITE FUNCTION
↑ WHOLE
↓ PART

$$\left[\ln x - \frac{1}{x} \right]_2^4$$

$$\left(\ln 4 - \frac{1}{4} \right) - \left(\ln 2 - \frac{1}{2} \right) = \ln 2 + \frac{1}{4}$$

$$\left[2 \sin \frac{1}{2}x \right]_{\pi/3}^{\pi}$$

$$1 \text{ u}^2$$

ROUTINE

PR. WHOLE
DER. PART

$$\int_0^1 e^{3x} dx$$

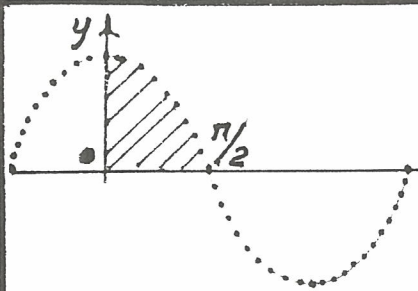
AREA BETWEEN $y = \cos x$

& x - y AXES

QUAD. I

F.F. PRIM. WHOLE
DER. PART $= \left[\frac{1}{3} e^{3x} \right]_0^1$

$$\frac{1}{3} e^3 - \frac{1}{3} = \frac{1}{3} (e^3 - 1) \text{ u}^2$$



$$1 \text{ u}^2$$

$$\left[\sin x \right]_0^{\pi/2}$$

INTEGRATION

93

$$\int_0^{\pi/2} \sec^2 \frac{1}{2} \theta \, dx$$

$$\int_0^{\pi/4} (1 + \tan^2 x) \, dx$$

F.F

$$\left[2 \tan \frac{1}{2} \theta \right]_0^{\pi/2} = 2 \, u^2$$

$$1 + \frac{\sin^2}{\cos^2} = \frac{1}{\cos^2} = \sec^2$$

$$\left[\tan x \right]_0^{\pi/4} = 1 \, u^2$$

$$\int_0^{\pi/2} \frac{2 \cos 2x}{1 + \sin 2x} \, dx$$

$$\int_1^K \frac{4}{x^2} \, dx = 3$$

ALREADY IN THE FORM
 $\frac{f'(x)}{f(x)}$

$$\left[\frac{-4}{x} \right]_1^K$$

$$\left[\ln(1 + \sin 2x) \right]_0^{\pi/2} = \ln 1.5$$

$\cdot 405 \, u^2$

$$\frac{-4}{K} + 4 = 3 \quad K = 4$$

$$\int_1^2 \frac{1}{x} \, dx$$

$$\int_0^{\pi/4} (\cos 2x) \, dx$$

$$\left[\ln x \right]_1^2 = \ln 2$$

F.F

$$\left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2}$$

INTEGRATION

94

$$\int_0^9 t^{\frac{1}{2}} dt$$

$$\int_1^2 \frac{1}{1+x} dx$$

$$\left[\frac{2}{\frac{3}{2}} t^{\frac{3}{2}} \right]_0^9 = 17\frac{1}{3} u^2$$

RECIPROCAL

$$\left[\frac{f'(x)}{f(x)} \ln(1+x) \right]_1^2 = \ln 1.5 u^2$$

$$\int_0^1 (3x^6 + 1) dx$$

$$\int_0^1 (1 + e^{-x}) dx$$

$$\left[\frac{3}{7} x^7 + x \right]_0^1 = 1\frac{3}{7} u^2$$

$$\left[x - e^{-x} \right]_0^1$$

$$1 - \frac{1}{e} + 1 = 2 - \frac{1}{e} u^2$$

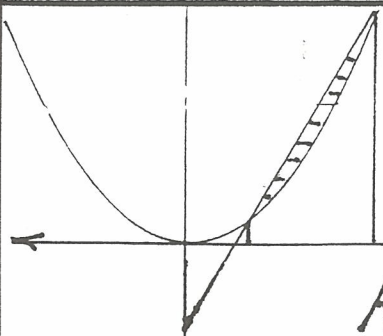
AREA BETWEEN

$$y = 3x - 2 \text{ \& } y = x^2$$

AT INTERSECTIONS

$$x^2 = 3x - 2$$

WHICH IS AT (1,1) \& (2,4)



AREA
AREA

$$\text{TRAPEZIUM} = 2\frac{1}{2} u^2$$

UNDER CURVE

$$\left[\frac{x^3}{3} \right]_1^2 = 2\frac{1}{3} u^2$$

REQUIRED

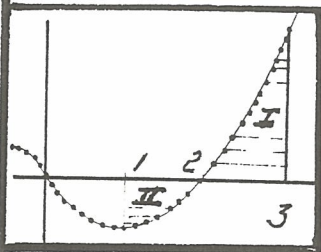
$$\text{AREA} = \frac{1}{6} u^2$$

INTEGRATION

95.

AREA BETWEEN

$y = x(x+1)(x-2)$, x -AXIS & ORDINATES
 $x = -1$ & $x = 3$



ROOTS AT $(0,0)$, $(-1,0)$, $(2,0)$

$$A_I = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_2^3 = 4 \frac{11}{12}$$

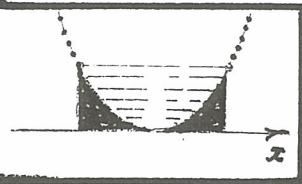
$$y = x^3 - x^2 - 2x$$

$$A_{II} = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^2 = \left| -\frac{7}{12} \right| \quad \text{MEANS BELOW } x\text{-AXIS}$$

TOTAL $6 \frac{1}{2} u^2$

AREA BETWEEN

$y = x^2$, x -AXIS & ORDINATES
 $x = \pm 2$

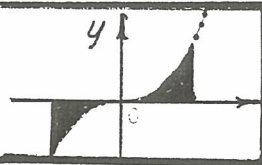


$$\left[\frac{x^3}{3} \right]_{-2}^2 \times 2$$

$$5 \frac{1}{3} u^2$$

AREA BETWEEN

$y = x^3$, x -AXIS & ORDINATES
 $x = \pm 2$



$$2 \left[\frac{x^4}{4} \right]_{-2}^2 = 8 u^2$$

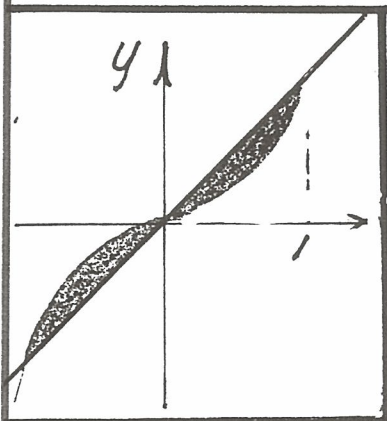
AREA RECTANGLE $= 16 u^2$
 AREA IN PARABOLA $= 10 \frac{2}{3} u^2$

AREAS ARE > 0
 BEWARE OF
 THOSE UNDER
 x -AXIS

INTEGRATION

96.

AREA BOUNDED
BY $y=x$ & $y=x^3$



INTERSECTIONS

$(-1, -1)$

$(0, 0)$

$(1, 1)$

$$A = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} u^2$$

AREA BOUNDED
BY $y=x$ & $y=x^3$

INTEGRATION ROUTINE
ALSO APPLIES TO
STRAIGHT LINES

$$\left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} = \frac{1}{2} BH = \frac{1}{2}$$

$$\text{AREA UNDER } x^3 = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

← SUBTRACT & DOUBLE
SAME

AREA BETWEEN
 $y=x^3$, y -AXIS, $y=0$ & 8

AREA BETWEEN
 $y=\frac{1}{x^2}$, y -AXIS, $y=1$ & 2

$$x = y^{\frac{1}{3}}$$

Y CONVERT
FIRST

$$\left[\frac{3}{4} y^{\frac{4}{3}} \right]_0^8 = 12 u^2$$

$$x = y^{-\frac{1}{2}}$$

Y CONVERT

$$\left[2y^{\frac{1}{2}} \right]_1^2$$

$$2\sqrt{2} - 2 \div 0.83 u^2$$

→ ALWAYS > 0

$x \neq 0$

INTEGRATION

97.

$$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{x^2-1} dx \quad \text{BEWARE!}$$

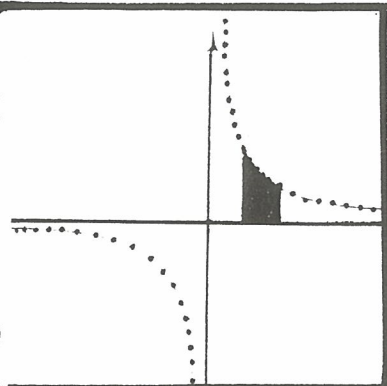
1. YOU MIGHT THINK LOG. $y = x^2, y' = 2x$ N.A.
2. YOU MIGHT THINK $(x^2-1)^{-1}$. No DIVISION BY 0!
3. YOU THEN RECOGNISE THE DIFFERENCE OF 2 SQUARES.

REWRITE AS

$$\frac{1}{2} \int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx = \left[\frac{\ln(x-1) - \ln(x+1)}{2} \right]_2^3 = \frac{1}{2} \ln 1.5$$

u^2

AREA UNDER
 $y = \frac{4}{x}$
 BETWEEN $x = 1 \frac{1}{2}$ & 2



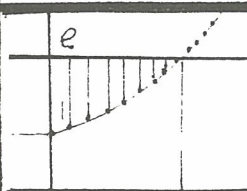
$$4 \left[\ln x \right]_1^2$$

$$4 \ln 2 \quad u^2$$

2.8

HYPERBOLA

AREA BETWEEN
 $y = e^x$, y-AXIS,
 $y = 1 \frac{1}{2}e$



MEANS $x = \ln y$.
 AVOID
 INTEGRATING $\ln y$.

4-UNIT

WHEN $y = e, x = 1$
 AREA RECTANGLE = $e \cdot u^2$
 AREA UNDER THE CURVE

$$\left[e^x \right]' = (e-1) u^2$$

REQUIRED $A = 1 u^2$

INTEGRATION

98.

DIFFERENTIATE
AND HENCE
INTEGRATE xe^x

DIFFERENTIATE
 $\ln \ln x$ AND HENCE PROVE
 $\int_{e^2}^{e^3} \frac{dx}{x \ln x} = \ln \frac{3}{2}$

PRODUCT RULE:

$$f'(xe^x) = e^x + xe^x$$

$$xe^x = f'(xe^x) - e^x$$

$$\int (xe^x) dx = xe^x - e^x + C$$

THIS PROCESS OF DERIVATION &
ANTI-DERIVATION IS USED WHEN
NORMAL ROUTINES DON'T APPLY.

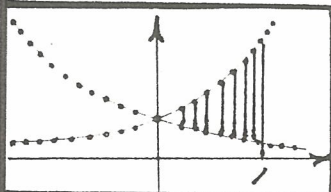
$$y' = \frac{1}{x \ln x}$$

$$\ln e^3 = 3$$

$$\left[\ln \ln x \right]_{e^2}^{e^3} = \ln 3 - \ln 2 = \ln \frac{3}{2}$$

AREA BETWEEN

e^{2x}, e^{-x} & ORDINATE
 $x=1$



EXPONENTIALS
PASS
THROUGH (0, 1)

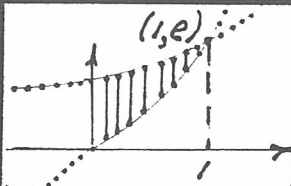
AREA BETWEEN

$x.e^{x^2}, e^x$ & y-AXIS

OFTEN, LIMITS HAVE TO BE
FOUND BY SOLVING
SIMULTANEOUS EQUATIONS

$$\left[\frac{1}{2} e^{2x} \right]' - \left[-\frac{1}{2} e^{-x} \right]'$$

$$\frac{1}{2} e^2 - \frac{1}{2} + \frac{1}{2} - 1 = 2.6 u^2$$



IF $y = e^{x^2}, y' = 2xe^{x^2}$

$$\int_0^1 x e^{x^2} = \left[\frac{e^{x^2}}{2} \right]_0^1$$

$$\left[e^x \right]' - \left[\frac{e^{x^2}}{2} \right]'$$

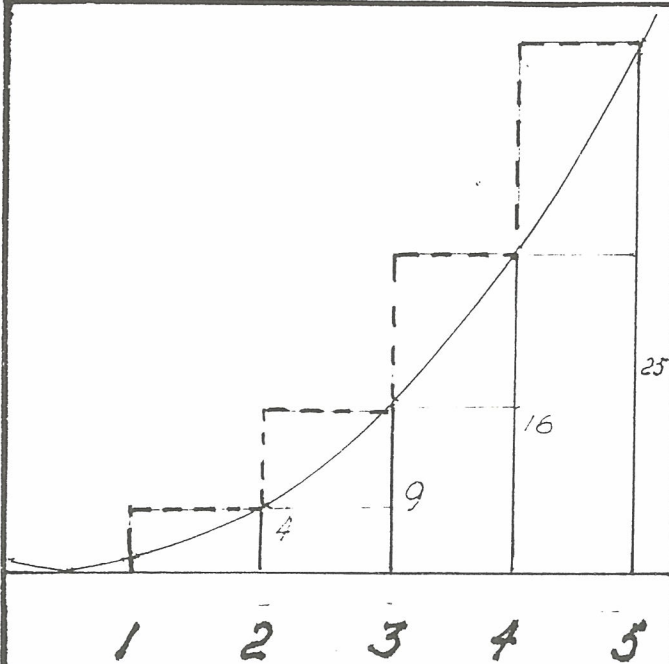
$$\frac{1}{2} e - \frac{1}{2} = .859 u^2$$

INTEGRATION

99.

APPROXIMATE AREA BOUNDED BY
 $y = x^2$, x -AXIS, ORDINATES $x = 1 \text{ \& } 5$

AVERAGE UPPER AND LOWER RECTANGLES



EXACT VALUE

WITH INTEGRATION

$$\left[\frac{x^3}{3} \right]_1^5$$

$$4\frac{1}{3} \text{ u}^2$$

1 INTERVAL

$$\frac{(4 \times 25) + (4 \times 1)}{2} = 52 \text{ u}^2$$

2 INTERVALS

$$\frac{(18+2) + (50+18)}{2} = 44 \text{ u}^2$$

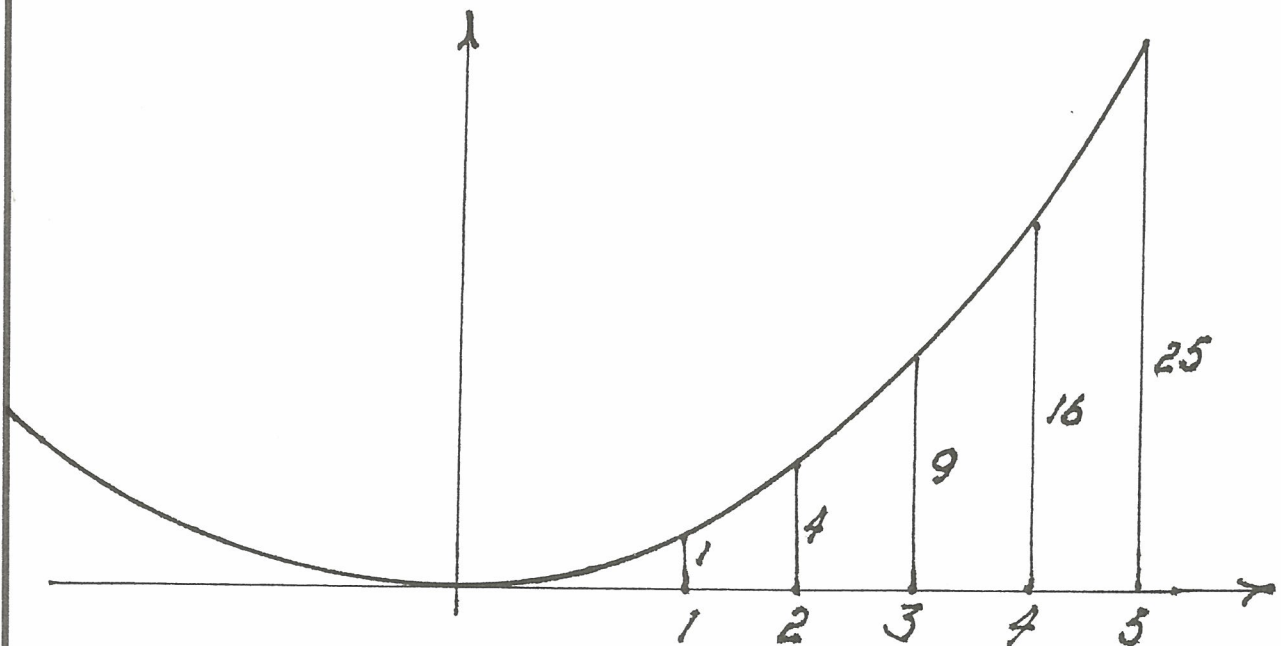
4 INTERVALS

$$\frac{(1+4+9+16) + (4+9+16+25)}{2} = 42\frac{1}{2} \text{ u}^2$$

INTEGRATION

100.

AREA UNDER $y=x^2$ BETWEEN $x=1$ & $x=5$
2 OTHER APPROXIMATION METHODS



MID - ORDINATE
RULE

4 INTERVALS

$$4 \times 2 + 16 \times 2 = 40 u^2$$

2 INTERVALS

$$9 \times 4 = 36 u^2$$

TRAPEZIUM RULE

$$A = \frac{1}{2} h (\text{SUM PARALLELS})$$

4 INTERVALS

$$\frac{1}{2} \{ (1+4) + (4+9) + (9+16) + (16+25) \}$$

$$42 u^2$$

2 INTERVALS

$$10 + 34 = 44 u^2$$

1 INTERVAL

$$2 \times 26 = 52 u^2$$

INTEGRATION

101.

AREA BETWEEN $y = x^2$, x-AXIS

$$x = 1 \text{ to } 5$$

EXTENDED SIMPSON RULE

EVEN NUMBER OF INTERVALS

EXACT FOR PARABOLAS

↓

SUB INTERVAL

3

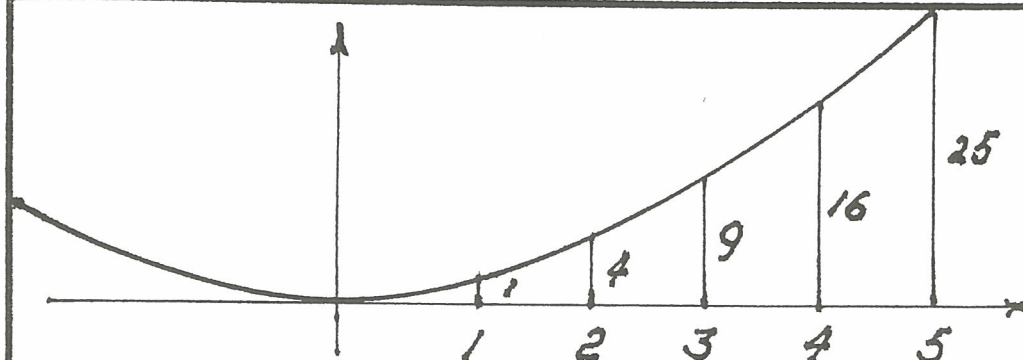
ORDINATES

(SUM ENDS + 2SUM OTHER ODDS + 4SUM EVENS)

HOW TO REMEMBER

NATURAL NUMBER ORDER: 1 2 (3) 4
ODDS ARE BEFORE EVENS!

THERE ARE 3 GROUPS



WITH
INTEGRATION

$$4\frac{1}{3} u^2$$

2 INTERVALS

3 ORDINATES

$$\frac{2}{3} (26 + 36) = 4\frac{1}{3} u^2$$

THERE ARE NO ODD ORDINATES IN POSITION

4 INTERVALS

5 ORDINATES

$$\frac{1}{3} (26 + 18 + 80) = 4\frac{1}{3} u^2$$

INTEGRATION

102.

PARABOLA, HEIGHT = 2m.
BASE FROM (-1,0) TO (1,0) = 2m

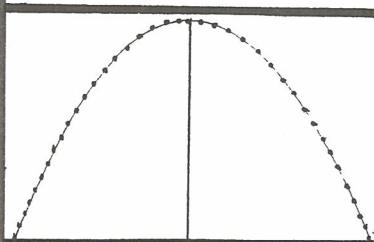
$$y = A(x-1)(x+1) \quad \text{SUB } (0,2) \quad \} A = -2$$

EQUATION

$$y = -2x^2 + 2$$

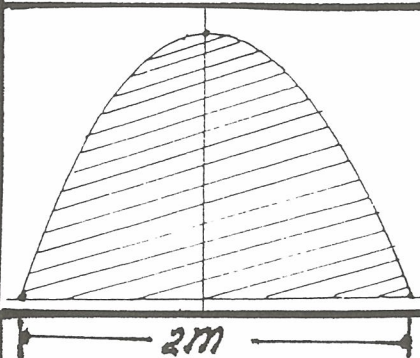
$$\left[\frac{-2x^3}{3} + 2x \right]_{-1}^1$$

$$A = 2 \frac{2}{3} \text{ m}^2$$



SIMPSON
 $\frac{1}{3} \times 8 = 2 \frac{2}{3} \text{ m}^2$

AREA UNDER $y = 2 \cos \frac{\pi}{2} x$
BETWEEN (-1,0) & (1,0)



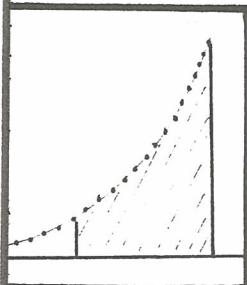
SIMPSON

$$\frac{1}{3} \times 8 \cos 0$$

$$\frac{8}{3} \text{ m}^2$$

$$2 \left[\frac{4 \sin \frac{\pi}{2} x}{\pi} \right]_0^1 = \frac{8}{\pi} \text{ m}^2$$

AREA UNDER $y = x^4$
 $x = 1 \text{ \& } 2$



$$\left[\frac{x^5}{5} \right]_1^2 = 6 \frac{1}{5} \text{ m}^2$$

AREA UNDER $\frac{1}{(1+3x)^2}$
 $x = 0 \text{ \& } \frac{1}{3}$

F.F. $\frac{\text{PRIM. WHOLE}}{\text{DER. PART}}$

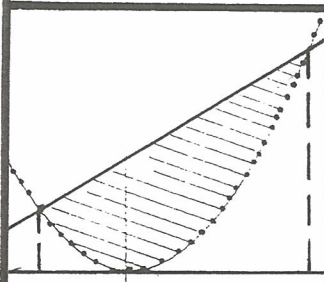
$$\left[\frac{(1+3x)^{-1}}{-3} \right]_0^{\frac{1}{3}} = \frac{4}{15} \text{ m}^2$$

AREA
BETWEEN

$$y = x+1 \text{ \& } y = 2x^2$$

AREA TRAPEZIUM

$$\frac{3}{4} \left(\frac{1}{2} + 2 \right) = \frac{15}{8}$$



INTERSECTIONS

$$(2x+1)(x-1) = 0$$

$$\left(-\frac{1}{2}, \frac{1}{2} \right) \text{ \& } (1, 2)$$

AREA UNDER CURVE

$$\left[\frac{2x^3}{3} \right]_{-\frac{1}{2}}^1 = \frac{3}{4}$$

$$\text{SHADED } A = \frac{9}{8} \text{ m}^2$$

INTEGRATION

103.

AREA UNDER $y = 5 + 2x - x^2$
 $x = -1 \text{ \& } 1$

AREA QUADRANT I
BETWEEN $y = x^3$ \& $y = x$

$$\left[5x + x^2 - \frac{x^3}{3} \right]_{-1}^1$$

CUBIC CURVE \& LINE
AT INTERSECTIONS
 $x = 0, x = 1$ ($x = -1, \text{N.A.}$)

$$9 \frac{1}{3} u^2$$

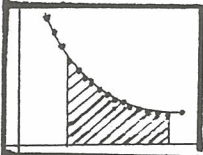
$$A = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} u^2$$

SPECIAL: AREA UNDER

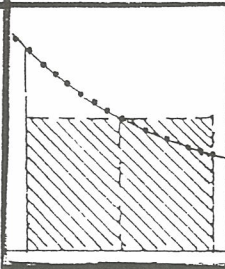
$$y = \frac{1}{x}, \quad x = 1 \text{ \& } x = b$$

MID-ORDINATE, 1 STRIP

$$\int_{3.7}^{4.3} \frac{1}{1 + \sqrt{x}} dx$$



$$A = [\ln x]_1^b = 1$$



WHEN $x = 4$,

$$M.O. = \frac{1}{3}$$

$$\ln b - \ln 1 = \ln b = 1$$

$$b = e$$

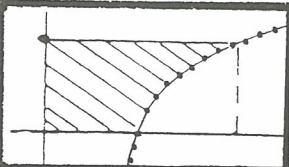
$$A \doteq \frac{1}{3} \times .6 = .2 u^2$$

AREA BETWEEN

$$y = \ln x, \quad x\text{-axis}, \quad y\text{-axis \& } y = \ln 2$$

AREA BETWEEN

$$y = x^3 - 3x^2 \text{ \& } x\text{-axis}$$



$$x\text{-INT} = (1, 0)$$

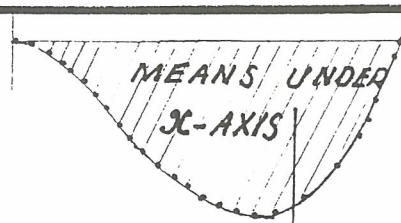
BECAUSE $e^0 = 1$

WHEN $\ln x = \ln 2, x = 2$

$y = \ln x$ MEANS $x = e^y$

$$[e^y]_0^{\ln 2} = 1 u^2$$

$$e^{\ln 2} = 2$$



MEANS UNDER
x-AXIS

$$x^2(x-3) = 0$$

INTERC.

$$(0, 0), (3, 0)$$

$$A = \left[\frac{x^4}{4} - x^3 \right]_0^3 = \left| -6 \frac{3}{4} \right| = 6 \frac{3}{4} u^2$$

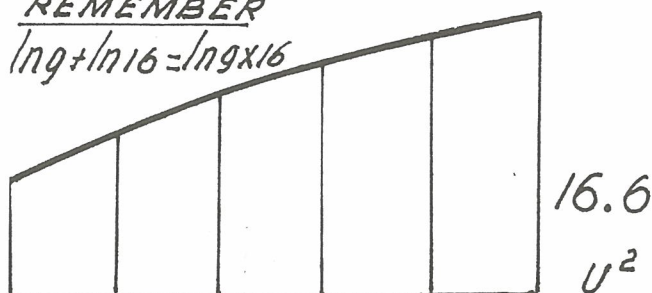
INTEGRATION

104.

$\int_3^8 \ln x^2 dx$ **TRAPEZOIDAL**
(TRAPEZIUM) **RULE**
5 INTERVALS

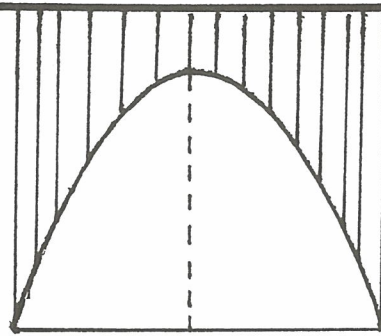
AREA BETWEEN
 $y=5$ & $y=4x-x^2$
POS. QUADRANT

REMEMBER
 $\ln 9 + \ln 16 = \ln 9 \times 16$



3 4 5 6 7 8

$\frac{1}{2} \{ \ln(9 \times 16^2 \times 25^2 \times 36^2 \times 49 \times 64) \}$



SIMPSON

$20 - \frac{2}{3} \times 16$
 $9\frac{1}{3} u^2$

0 2 4

$A = 20 - \left[2x^2 - \frac{x^3}{3} \right]_0^4 = 9\frac{1}{3} u^2$

AREA BETWEEN

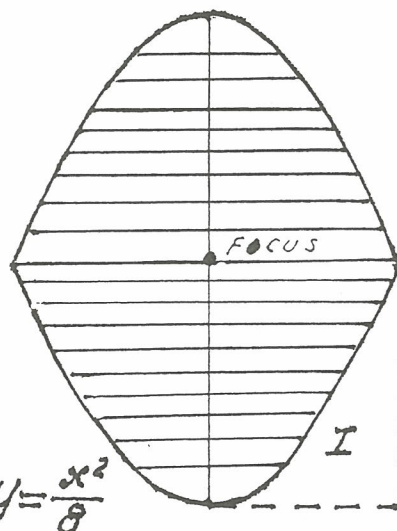
$x^2 = 8(4-y)$ & $x^2 = 8y$

$x^2 = -8(y-4)$

VERTEX (0, 4)
FOCUS (0, 2)
FOCAL LENGTH = 2
DIRECTRIX $y=6$

INTERSECTIONS

$8y = 32 - 8y$
 $(-4, 2) \text{ \& } (4, 2)$



VERTEX (0, 0)
FOCUS (0, 2)
FOCAL LENGTH = 2
DIRECTRIX $y=-2$

$y = \frac{x^2}{8}$

RECTANGLE = $8 u^2$

AREA I = $\left[\frac{x^3}{24} \right]_0^4 = \frac{8}{3} u^2$

TOTAL AREA = $4 \times \frac{16}{3} = 21\frac{1}{3} u^2$

INTEGRATION

105

$$f(x) = x/\ln x$$

$$\int_1^5 x/\ln x dx$$

5 VALUES
USING
SIMPSON

$$\int_{-\pi/3}^{\pi/3} \cos^2 x dx$$

x	1	2	3	4	5
f(x)	0	1.39	3.3	5.55	8.05

SIMPSON
3 VALUES
 $\pi/3 = 60^\circ$
 $\cos 0 = 1$
 $\cos 60 = 1/2$

$$A \doteq \frac{1}{3} (8.05 + 6.6 + 4 \times 6.94)$$

$$14.14 U^2$$

$$A = \frac{\pi}{9} \left(\frac{1}{2} + 4 \right) = \frac{\pi}{2} U^2$$

$$\int_0^1 4^x dx$$

SIMPSON
3 VALUES

TRAPEZOIDAL

$$f(x) = e^{2x} (1-x)$$

$$A \doteq \frac{1}{6} (5+8) = 2 \frac{1}{6} U^2$$

x	-3	-2	-1	0	1
f(x)	.01	.05	.27	1	0

$$A \doteq \frac{1}{2} \{ .01 + 2(.05 + .27 + 1) \} = 1.33 U^2$$

UNDER ESTIMATE

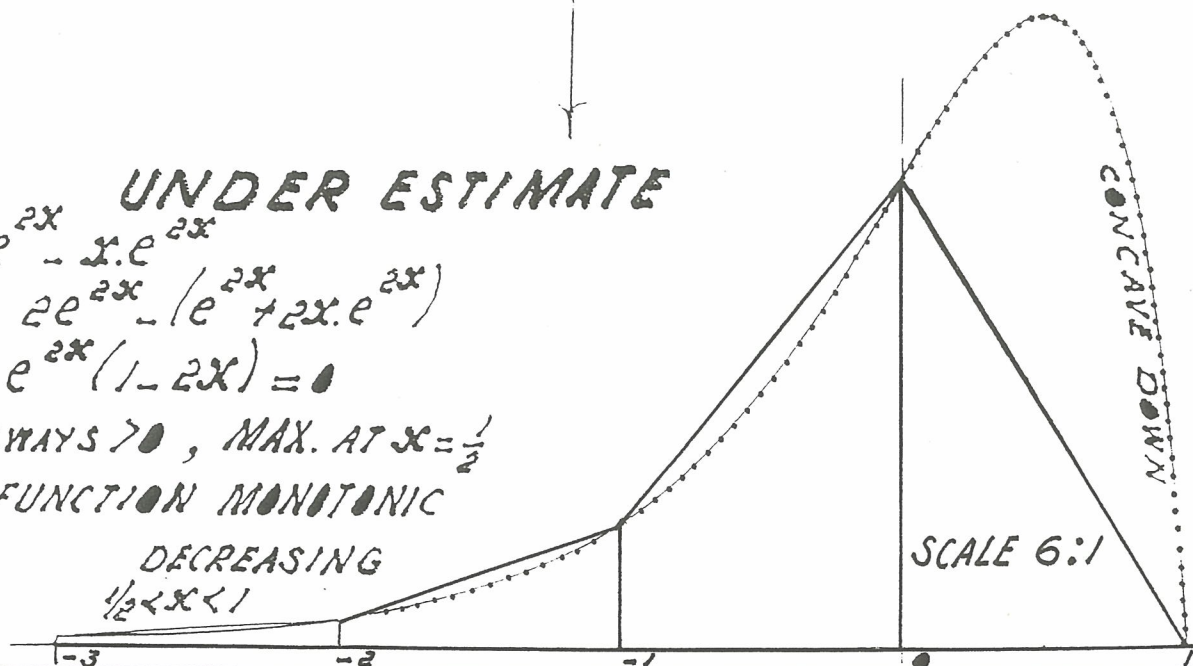
$$y = e^{2x} - x \cdot e^{2x}$$

$$y' = 2e^{2x} - (e^{2x} + 2x \cdot e^{2x})$$

$$= e^{2x} (1 - 2x) = 0$$

 e^{2x} ALWAYS > 0 , MAX. AT $x = \frac{1}{2}$

FUNCTION MONOTONIC

DECREASING
 $\frac{1}{2} < x < 1$


INTEGRATION

106.

WHEN AN AREA IS ROTATED
ABOUT THE x -AXIS OR
THE y -AXIS,

THE VOLUME OF A SOLID OF REVOLUTION
IS GENERATED

A RECTANGLE

GENERATES

A CYLINDER

A SEMI-CIRCLE

GENERATES

A SPHERE

A RAY FROM $(0,0)$

GENERATES

A CONE

ROTATION ABOUT x -AXIS $x=a$ to b

LET ΔV BE A SMALL INCREMENT AT $x=b$
AS Δx APPROACHES 0, ΔV APPROACHES A CIRCLE

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = \frac{dV}{dx} = V' = \pi y_b^2 \quad R = y_b$$

$$V_a = \pi y_a^2 + C = 0$$

$$C = -\pi y_a^2$$

$$V_b = \pi (y_b^2 - y_a^2)$$

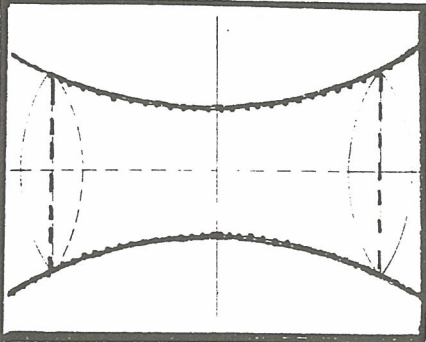
TO REMEMBER
SAY

$$V = P_1 y^2$$

ROTATION ABOUT

X-AXIS: $y = x^2 + 1$

BETWEEN $x = -1$ & 1



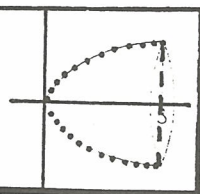
$$y^2 = (x^4 + 2x^2 + 1)$$

$$2\pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_{-1}^1 = \frac{56\pi}{15} \text{ units}^3$$

ROTATION ABOUT

X-AXIS: $y = \sqrt{x}$

BETWEEN $x = 0$ & 3



$$y^2 = x$$

GENERATED VOLUME

$$\pi \left[\frac{x^2}{2} \right]_0^3 = \frac{9\pi}{2} \text{ units}^3$$

ROTATION ABOUT

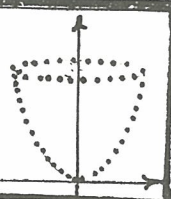
Y-AXIS: $y = x^2$ ALREADY
BETWEEN $y = 0$ & 3

ROTATION ABOUT

Y-AXIS: $y = x^3$
BETWEEN $y = 1$ & 8

CONVERT TO

$$x = y^{1/3}$$



$$V = \pi \text{ EX SQUARE}$$

$$V = \pi \left[\frac{y^2}{2} \right]_0^3 = \frac{9\pi}{2} \text{ units}^3$$

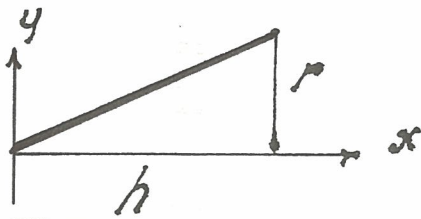
$$x^2 = y^{2/3}$$

$$\pi \left[\frac{3}{5} y^{5/3} \right]_1^8 = 58.43 \text{ units}^3$$

INTEGRATION

108.

CONFIRMING THE VOLUME OF A CONE



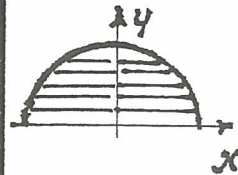
ROTATION ABOUT
X-AXIS

$$y = \frac{r}{h}x, \text{ BETWEEN } x=0 \text{ \& } h$$

$$\pi \frac{r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h \quad \text{GENERATED } V$$

$$\frac{1}{3} r^2 \pi \cdot h$$

CONFIRMING THE VOLUME OF A SPHERE



ROTATION ABOUT
X-AXIS: $y = \sqrt{R^2 - x^2}$
BETWEEN $x = -R$ \& R

$$y^2 = (R^2 - x^2)$$

$$2\pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R \quad \text{GENERATED } V$$

$$\frac{4}{3} R^3 \pi$$

ROTATION ABOUT X-AXIS: REGION

BETWEEN $y = x^2$ \& $x = y^2$

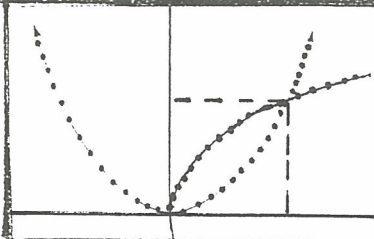
ROTATION ABOUT Y-AXIS: REGION

BETWEEN $y = x^2$ \& $x = y^2$

AT INTERSECTION

$$x^4 = x$$

$$x = x$$



$$x(x^3 - 1) = 0$$

$$(0,0) \text{ \& } (1,1)$$

$$A = \int_0^1 (\sqrt{x} - x^2) dx$$

$$\left[\frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{3} \text{ S.U.}$$

GENERATED VOLUME

$$\pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{3\pi}{10} \text{ U}^3$$

INTER CHANGE x \& y

GENERATED VOLUME

$$\pi \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 = \frac{3\pi}{10} \text{ U}^3$$

INTEGRATION

109

ROTATION ABOUT X-AXIS: $y = x^2 + 1$, $x = -1 \text{ to } 1$

$$y^2 = x^4 + 2x^2 + 1$$

$$V = 2\pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_{-1}^1 = 11.73 \text{ u}^3$$

INSTEAD OF INTEGRATING,
THIS PART CAN BE REPLACED
BY USING SIMPSON WITH

$$y^2 = x^4 + 2x^2 + 1$$

AND 3 FUNCTION VALUES
 $x = 0, x = .5, x = 1$

KEEP!! $\times \frac{1}{6}$

$$\frac{\pi}{3} (1 + 4 + 4 \left(\frac{1}{16} + \frac{1}{2} + 1 \right)) = 11.78 \text{ u}^3$$

ROTATION ABOUT Y-AXIS: BETWEEN $y = x^2 - 3$
AND X-AXIS

$$\frac{x^2 = y + 3}{\text{EXACT VALUE WITH INTEGRATION}}$$

$$V = \pi \left[\frac{y^2}{2} + 3y \right]_{-3}^0 = 14.14 \text{ u}^3$$

WITH SIMPSON

$$y = 0, y = -1.5, y = -3$$

$$x^2 = y + 3$$

GENERATED VOLUME
KEEP!

$$\frac{1.5\pi}{3} (3 + 6) = 14.14 \text{ u}^3$$

ROTATION ABOUT X-AXIS
 $y = \sqrt{x}$, $x = 0 \text{ to } 3$

$$y^2 = x$$

$$V = \pi \left[\frac{x^2}{2} \right]_0^3 = 14.14 \text{ u}^3$$

SIMPSON: $y^2 = x$ (2INT)

$$V = \frac{1.5\pi}{3} (3 + 6) = 14.14 \text{ u}^3$$

KEEP THE π !

ROTATION ABOUT Y-AXIS
 $y = 1 \text{ to } 8$ $y = x^3$

$$x^2 = y^{2/3}$$

$$V = \pi \left[\frac{3}{5} y^{5/3} \right]_1^8 = 58.43 \text{ u}^3$$

SIMPSON: $x^2 = y^{2/3}$ (2INT)

$$V = \frac{3.5\pi}{3} (1 + 4 + 4 \times 4.5^{2/3}) = 58.29 \text{ u}^3$$

KEEP THE π !

INTEGRATION

110.

ROTATION ABOUT

X-AXIS: $y = \sec x$

$$0 \leq x \leq \frac{\pi}{4}$$

GENERATED VOLUME

$$\pi \left[\tan x \right]_0^{\pi/4} = \pi u^3$$

ROTATION ABOUT

Y-AXIS: REGION

BETWEEN $y = 4 - x^2$ &
X-AXIS

ALSO ABOUT X-AXIS

$$x^2 = 4 - y$$

$$V = \pi \left[4y - \frac{y^2}{2} \right]_0^4 = 8\pi$$

$$y^2 = 16 - 8x^2 + x^4$$

$$V = 2\pi \left[16x - \frac{18x^3}{3} + \frac{x^5}{5} \right]_0^4 = \frac{512}{15} \pi$$

ROTATION ABOUT

X-AXIS: $y = e^x$

$$1 \leq x \leq 1.5_{3sf} \quad 0 \leq x \leq 2$$

$$V = \frac{\pi}{2} \left[e^{2x} \right]_1^{1.5} = 19.9 u^3$$

$$V = \frac{\pi}{2} (e^4 - 1) u^3 = 84.2 u^3$$

ROTATION ABOUT

X-AXIS: $x^2 + 4y^2 = 16$

$$x = 0, y = \pm 2$$

$$y = 0, x = \pm 4$$

ELLIPSE

$$y^2 = \frac{1}{4}(16 - x^2)$$

$$V = \frac{\pi}{2} \left[16x - \frac{x^3}{3} \right]_0^4 = \frac{64\pi}{3} u^3$$

ROTATION ABOUT X-AXIS

$y = 2x - x^2$, BETWEEN $x = 0$ & 2

$$y^2 = x^4 - 4x^3 + 4x^2$$

$$V = \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^2 = \frac{16\pi}{15} u^3$$

ROTATION ABOUT X-AXIS

$y = \frac{1}{\sqrt{1+x^2}}$, BETWEEN $x = 3$ & 0

$$y^2 = \frac{1}{x^2+1} = \frac{f'(x)}{f(x)}$$

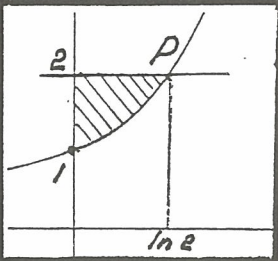
$$V = \pi \left[\ln(x+1) \right]_0^3 = \pi \ln 4 u^3$$

INTEGRATION

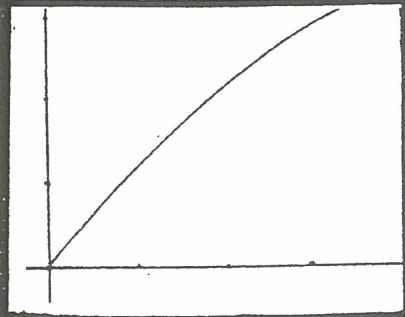
III.

AREA BETWEEN
 $y=e^x$, $y=2$ & y -AXIS

ROTATION ABOUT
 y -AXIS: $y=\ln(1+x)$ $x=0$
 $x=1$



AT P
 $e^x = 2$
MEANS
 $x = \ln 2$



CONVERSION

y
 $e^y = 1+x$
 $x = e^y - 1$

AREA RECTANGLE $= 2 \ln 2 \text{ u}^2$
AREA UNDER CURVE
 $[e^x]_0^{\ln 2} = 1 \text{ u}^2$
REQUIRED AREA: $(2 \ln 2 - 1) \text{ u}^2$

$x^2 = e^{2y} - 2e^y + 1$

$V = \pi \left[\frac{e^{2y}}{2} - 2e^y + y \right]_0^{\ln 2}$

$e^{\ln 2} = 2$
 $e^{2 \ln 2} = 4$
 $\pi \left(\ln 2 - \frac{1}{2} \right) \text{ u}^3$

ROTATED ABOUT
 y -AXIS
SIMPSON (2INT.)

ROTATION ABOUT
 x -AXIS: $y = \sqrt{9-x^2}$

CONVERT

$y = e^x$ MEANS
 $x = \ln y$
 $x^2 = (\ln y)^2$

FROM SEMI-CIRCLE TO SPHERE

$y^2 = 9 - x^2$

$V = 2\pi \left[9x - \frac{x^3}{3} \right]_0^3 = 36\pi \text{ u}^3$
WHICH IS $\frac{4}{3}\pi R^3$

3 FUNCTION VALUES: $f(2), f(1.5), f(1)$

2 INTERVALS: SIMPSON
EXTENDED

$V = \frac{\pi}{6} \{ (\ln 1)^2 + (\ln 2)^2 + 4(\ln 1.5)^2 \}$
 $.596 \text{ u}^3$

$36\pi \text{ u}^3$

INTEGRATION

112

ROTATION ABOUT

X-AXIS: $y = 2(1 + e^{2x})$
 $x=0$ $x=1/2$

$$y^2 = 4 + 8e^{2x} + 4e^{4x}$$

$$V = \pi \left[4x + 4e^{2x} + e^{4x} \right]_{x=0}^{x=1/2}$$

FUNCTION OF A FUNCTION (COMPOSITE)
 PRIM. WHOLE ÷ DER. PART

$$\pi (e^2 + 4e - 3) u^3$$

ROTATION ABOUT

X-AXIS: $y = x + \frac{1}{x}$, $x=1$ $x=3$

$$y^2 = x^2 + 2 + x^{-2}$$

$$V = \pi \left[\frac{x^3}{3} + 2x - \frac{1}{x} \right]_{x=1}^{x=3}$$

$$\frac{40\pi}{3} u^3 (41.9)$$

$$\frac{\pi}{3} (4 + 11\frac{1}{9} + 25) = 42 u^3 \text{ SIMPSON}$$

ROTATION ABOUT

X-AXIS: $y = \sqrt{x} - 1$, $x=2$ $x=3$

$$y^2 = x - 2x^{1/2} + 1$$

$$V = \pi \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} + x \right]_2^3$$

$$\pi \left\{ \left(\frac{9}{2} - 4\sqrt{3} + 3 \right) - \left(4 - \frac{8}{3}\sqrt{2} \right) \right\}$$

$$1.078 u^3$$

ROTATION ABOUT

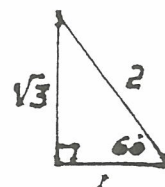
X-AXIS: $y = \sec x$, $0 \leq x \leq \frac{\pi}{3}$

$$y^2 = \sec^2 x$$

$$V = \pi \left[\tan x \right]_0^{\pi/3}$$

$$\pi \sqrt{3} \text{ units}^3$$

5.4



INTERSECTION SIMULTANEOUS EQUATION

113

$$-3x + y = 3$$

$$3x - 2y = 8$$



TYPE 1

ADD TO

ELIMINATE x

$$y = -11$$

SUBSTITUTE

$$x = -4\frac{2}{3}$$

$$7x + 5y = 23$$

$$2x + 5y = 3$$



TYPE 2

SUBTRACT

TO ELIMINATE y

$$x = 4$$

$$y = -1$$

$$x + y = 1$$

$$3x - 2y = 8$$



TYPE 3

ADJUST TO

ELIMINATE y (OPP. SIGNS)

$$\begin{array}{r} 2x + 2y = 2 \\ + \quad 3x - 2y = 8 \\ \hline \end{array}$$

MENTALLY $x = 2$

$$y = -1$$

$$2x - 3y = 5$$

$$5x + 2y = -16$$



TYPE 4

ADJUST BOTH

TO ELIMINATE y

$$\begin{array}{r} 4x - 6y = 10 \\ + \quad 15x + 6y = -48 \\ \hline \end{array}$$

MENTALLY $x = -2$

$$y = -3$$

INTERSECTION SIMULTANEOUS EQUATIONS 114

$$x + y + z = 6 \quad (1)$$

$$2x + 3y + z = 13 \quad (2)$$

$$x + 2y - z = 5 \quad (3)$$

$$(2) + (3)$$

$$3x + 5y = 18$$

$$(1) + (3)$$

$$2x + 3y = 11$$

ADJUST BOTH

$$6x + 10y = 36$$

$$6x + 9y = 33$$

$$y = 3$$

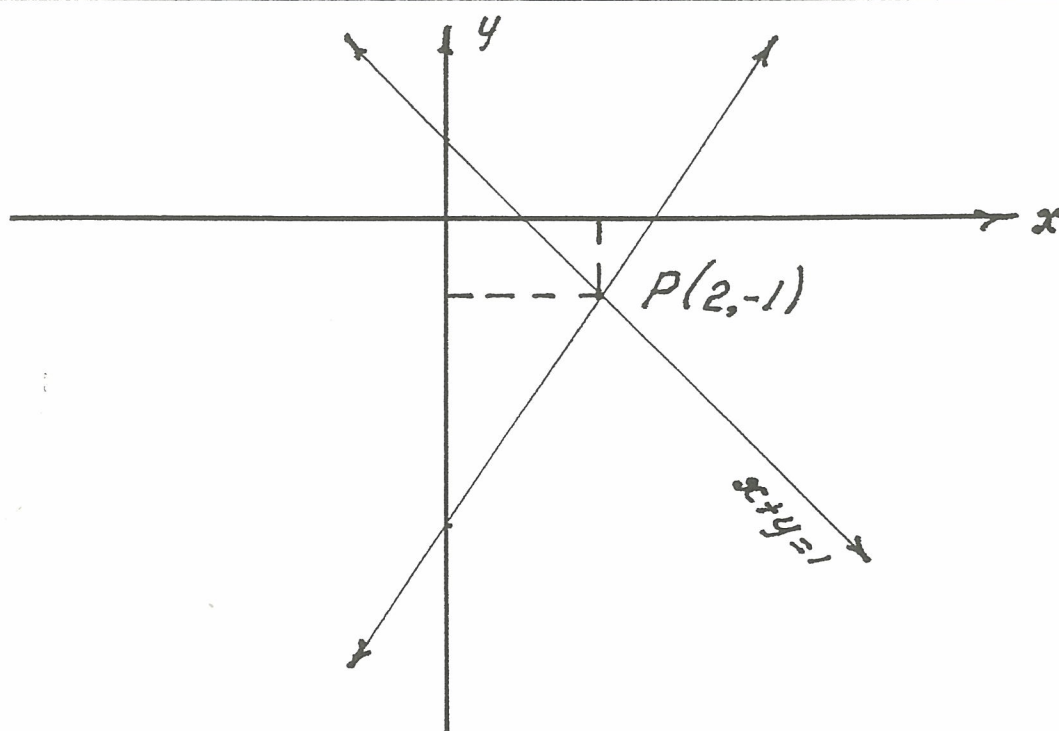
$$x = 1$$

$$z = 2$$

$$x + y = 1$$

$$3x - 2y = 8$$

GRAPHIC SOLUTION



INTERSECTION SIMULTANEOUS EQUATIONS

115.

LINE & PARABOLA

$$y = 2x + 3 \quad y = x^2 + 4x - 5$$

$$x^2 + 4x - 5 = 2x + 3$$

$$x^2 + 2x - 8 = 0$$

$$(x+4)(x-2) = 0$$

INTERSECTIONS AT

$$(-4, -5) \text{ \& } (2, 7)$$

2 LINES, TEST 3RD

$$4x - 3y - 12 = 0$$

$$8x + 3y - 6 = 0$$

$$\text{IS } y = 2x - 2$$

CONCURRENT

$$12x = 18$$

SUB.

P
FALSE

NOT

CONCURRENT

together run

INTERSECTION
AT

$$P(1\frac{1}{2}, -2)$$

LINE & HYPERBOLA

$$x+1 \cap \frac{2}{x}$$

$$x+1 = \frac{2}{x}$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

INTERSECTIONS AT

$$(-2, -1) \text{ \& } (1, 2)$$

LINE & CUBIC CURVE

$$y = 0$$

$$y = x^3 - x^2 - 6x$$

$$x(x^2 - x - 6) = 0$$

$$x(x-3)(x+2) = 0$$

INTERSECTIONS AT

$$(0, 0), (3, 0) \text{ \& } (-2, 0)$$

INTERSECTION SIMULTANEOUS EQUATIONS

116.

LINE &	CIRCLE	
$y = x + 2$	$x^2 + y^2 = 16$	$y = \cos x \text{ \& } y = \sin x$ $0 \leq x \leq \pi/2$
$x^2 + (x+2)^2 = 16$ $2x^2 + 4x + 4 = 16$ $x^2 + 2x - 6 = 0$ $x = -1 \pm \sqrt{7}$ <u>INTERSECTIONS AT</u> $(-1-\sqrt{7}, 1-\sqrt{7}) \text{ \& } (-1+\sqrt{7}, 1+\sqrt{7})$		$\sin 90-x = \sin x$ $x = 45^\circ$ <u>INTERSECTION AT</u> $(\pi/4, 1/\sqrt{2})$
$y^2 = x^2 - 21$	$x + y = 3$	<u>PARABOLA & CIRCLE</u> $y = x^2$ $x^2 + y^2 = 9$
$y = 3 - x$ $y^2 = 9 - 6x + x^2$ $x^2 - 21 = 9 - 6x + x^2$ $6x = 30$ <u>INTERSECTION AT</u> $(5, -2)$		$y^2 + y - 9 = 0$ $y = \frac{-1 \pm \sqrt{37}}{2}$ <u>$y > 0$ ONLY</u> $y \approx 2.5$ <u>APPROXIMATE INTERSECTIONS</u> <u>AT</u> $(1.6, 2.5)$ $(-1.6, 2.5)$

INTERSECTION/SIMULTANEOUS EQUATIONS

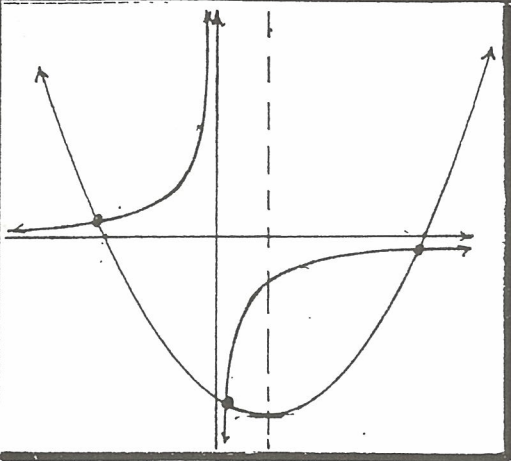
117

PARABOLA & HYPERBOLA

$$y = x^2 - 2x - 3$$

$$xy = -1$$

WRITE AN EQUATION THE ROOTS OF WHICH ARE INTERS. ABSCISSAE



$$x^2 - 2x - 3 = \frac{-1}{x}$$

$$x^3 - 2x^2 - 3x + 1 = 0$$

2 PARABOLAS

$$y = 3x^2 - 4x + 5$$

$$y = 2x^2 + 3x - 5$$

$$3x^2 - 4x + 5 = 2x^2 + 3x - 5$$

$$x^2 - 7x + 10 = 0$$

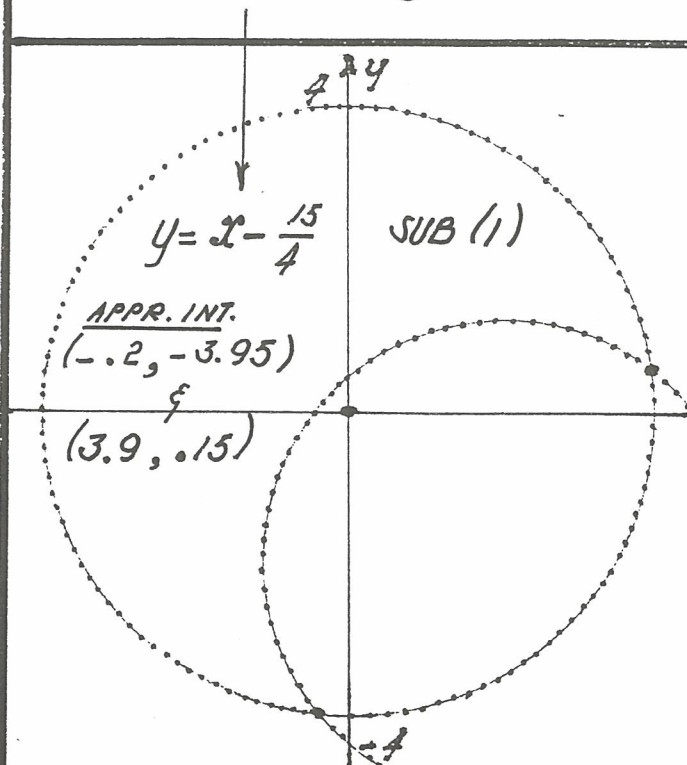
$$(x - 5)(x - 2) = 0$$

INTERSECTIONS AT
(5, 60) & (2, 9)

2 CIRCLES

$$(1) \quad x^2 + y^2 = 16$$

$$(x - 2)^2 + (y + 2)^2 = 9$$



CIRCLE & HYPERBOLA

$$x^2 + y^2 = 16$$

$$xy = 2$$

$$x^2 + \frac{4}{x^2} = 16$$

$$x^4 - 16x^2 + 4 = 0$$

$$x^2 = 15.7$$

$$x^2 = .25$$

APPR. INTERSECTIONS AT:

(.5, 3.96), (3.96, .5), (-.5, -3.96),
& (-3.96, -.5)

INTERSECTION SIMULTANEOUS EQUATIONS 118.

$$y = mx \text{ \& } y = x^3 - 4x^2 + 3x$$

INTERSECT AT (0,0)

WHEN $m < -1$ NO OTHER POINT
IN COMMON. DISCUSS $m = -1$

$$y = ax^2 - c \text{ \& } x^2 + y^2 = 16$$

MEET ON BOTH X- AND
Y AXES. Q70 C70

$$x \{ x^2 - 4x + (3-m) \} = 0$$

$$x = 0$$

NO OTHER VALUES WHEN
 $16 - 12 + 4m < 0$
WHEN $m < -1$

$$\text{SUB}(0, -4): c = 4$$

$$\text{SUB}(4, 0) \text{ OR } (-4, 0)$$

$$16a - 4 = 0$$

$$m = -1: x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0$$

$y = -x$ PASSES THROUGH
(0,0) AND IS A TANGENT AT
(2, -2)

$$a = \frac{1}{4}$$

$$\text{DO } y = 3x^2 - 4x + 5$$

$$\text{\& } y = 2x^2 + 3x - 4$$

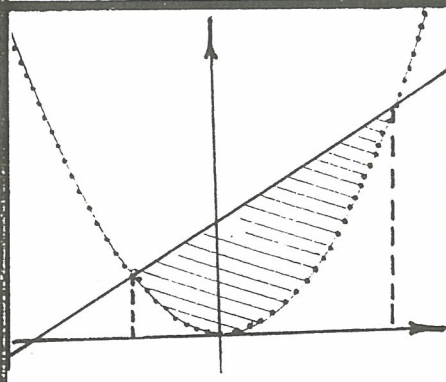
CROSS X-AXIS?

$$y = x+1 \text{ \& } y = 2x^2$$

$$y = 3x^2 - 4x + 5$$

$$16 - 60 < 0$$

PARABOLA CONCAVE UP.
POSITIVE DEFINITE,
NO X-INTERCEPTS.



$$\text{AREA}$$

$$\frac{15}{8} \text{ (TRAP.)}$$

$$-\left[\frac{2}{3}x^3\right]_{-1}^{2}$$

$$= 1\frac{1}{8} \text{ U}^2$$

$$y = 2x^2 + 3x - 4$$

$$9 + 32 > 0$$

PARABOLA CONCAVE UP
INDEFINITE, 2 X-INTERCEPTS

$$2x^2 - x - 1 = 0$$

$$\begin{matrix} (2x & +1) \\ (x & -1) \end{matrix}$$

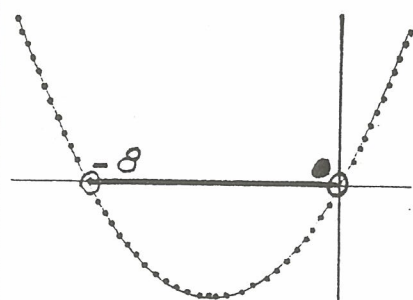
INTERSECTIONS
AT $(-\frac{1}{2}, \frac{1}{2})$ \&

$$(1, 2)$$

INTERSECTION SIMULTANEOUS EQUATIONS 119.

FOR WHAT VALUES OF m
DOES $y = m(x+1)$ HAVE
NO INTERSECTIONS WITH
 $y = 2x^2$

EQUATIONS TWO
TANGENTS TO $y = 2x^2$
WHICH PASS THROUGH
 $(-1, 0)$



$y = m(m+8)$
 < 0
BELOW
X-AXIS

$$y = mx + b$$

SUB $(-1, 0)$

$$0 = -m + b \therefore m = b$$

$\therefore$ LINE $y = m(x+1)$ PASSES
THROUGH $(-1, 0)$. TO BE A
TANGENT, $2x^2 - m(x+1) = 0$
MUST HAVE EQUAL ROOTS

$$m(m+8) = 0$$

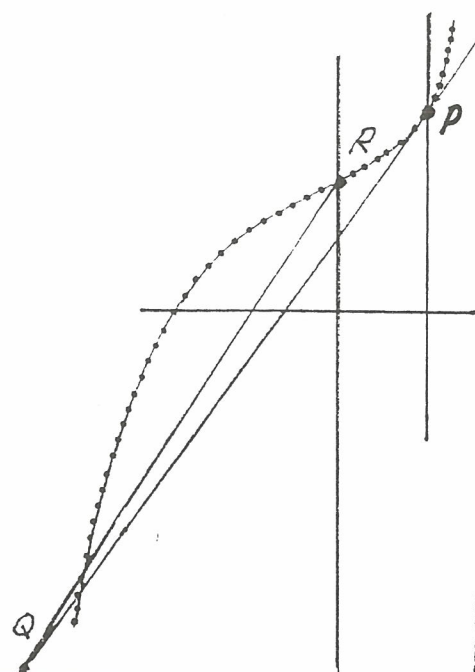
$$m = 0 \text{ \& } m = -8$$

EQUATIONS: $y = 0$ \& $y = -8(x+1)$

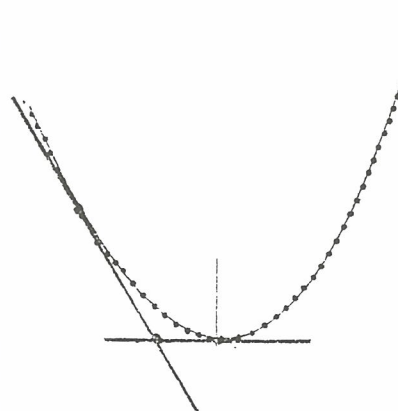
$2x^2 - mx - m = 0$
NO INTERSECTION (ROOTS)
WHEN $m(m+8) < 0$
WHEN
 $-8 < m < 0$ TEST $m = -4$, TRUE

$y = x^3$ \& LINES
 $y = x + 5$, $x = 0$ \& $x = 1$
EQ. TANGENT AT P.
 $Q(-5, -12)$ EQ. QR

POINT AT WHICH
 $y = -8x - 8$ IS A
TANGENT TO $y = 2x^2$



$P(1, 6)$
 $y' = 3x^2 = 3$
 $6 = 3 + b$
 $y = 3x + 3$
SUB Q ✓
 $R(0, 5)$
 $m_{QR} = \frac{17}{5}$
 $y = \frac{17}{5}x + 5$



$x^2 + 4x + 4 = 0$
 $(x+2)^2 = 0$
TANGENT
AT
 $(-2, -8)$

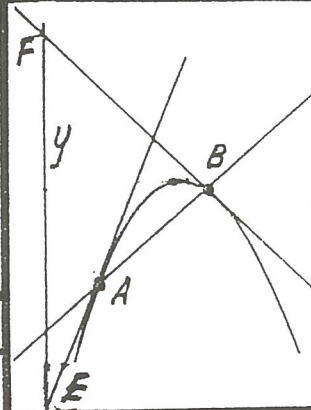
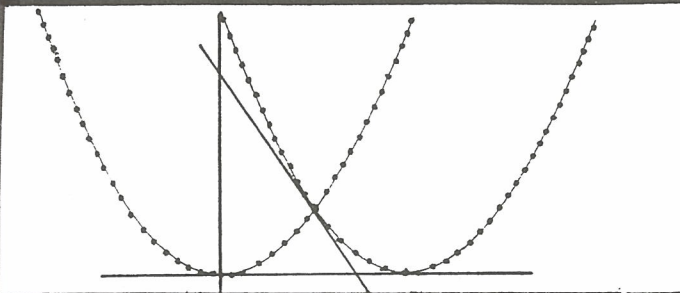
GRADIENT LINE = -8

CHECK: $y' = 4x = -8$

INTERSECTION SIMULTANEOUS EQUATIONS 120.

SHOW $P(2,1)$ ON $4y = x^2$ &
 $4y = (x-4)^2$ & THE ONLY POINT
 EQ. TANGENT AT P TO $4y = (x-4)^2$
 & ITS INT. Q WITH $4y = x^2$

$y = x+3$ & $y = 5x - x^2$
 Y-INT. TANGENTS AT INT. A & B
 ARE E & F
 SHOW $EF = 8$ UNITS



$$\begin{aligned} x+3 &= 5x-x^2 \\ x^2-4x+3 &= 0 \\ (x-3)(x-1) &= 0 \\ A(1,4) \quad B(3,6) \\ y' &= 5-2x \end{aligned}$$

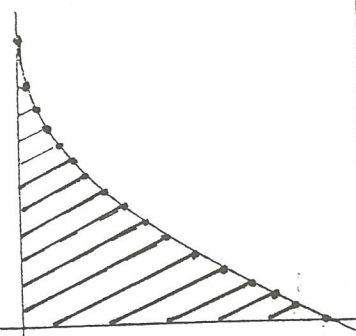
SUB (2,1) IN BOTH. TRUE
 $(x-4)^2 = x^2$, $x=2$ TRUE

2 PARABOLAS: FOCAL LENGTH 1
 VERTICES $(0,0)$ & $(4,0)$. SAME
 SIZE.
 $y = \frac{1}{4}(x-4)^2$. $y' = \frac{1}{2}(x-4) = -1$ AT P
 EQ. $y = -x+3 = \frac{x^2}{4}$. $Q(-6,9)$

$$\begin{aligned} m_A &= 3 & m_B &= -1 \\ 4 &= 3+b & 6 &= -3+b \\ y &= 3x+1 & y &= -x+9 \\ E &\leftarrow 8 \text{ UNITS} \rightarrow F \end{aligned}$$

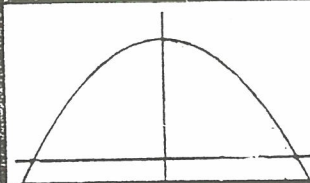
X- & Y-INT. $y = 4 - \sqrt{2x}$
 ROTATION
 ABOUT Y-AXIS

X-INTERCEPTS
 $y = 1 - x^2$
 AREA UNDER CURVE
 ROTATION ABOUT X-AXIS



CURVE ONLY FOR
 $x \geq 0$

X-INT. $(8,0)$
 Y-INT. $(0,4)$



X-INTERCEPTS

$(-1,0)$ & $(1,0)$

$$A = 2 \left[x - \frac{x^3}{3} \right]' = \frac{1}{3} u^2$$

$$\sqrt{2x} = 4 - y$$

$$2x = (4-y)^2$$

$$x = \frac{1}{4}(4-y)^2$$

$$V = \frac{\pi}{2} \left[(4-y)^5 \right]' = \frac{5}{2} \pi u^3$$

$$y^2 = 1 - 2x^2 + x^4$$

$$\begin{aligned} V &= 2\pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]' \\ &= \frac{16\pi}{15} u^3 \end{aligned}$$

LIMITS

121.

$$\lim_{x \rightarrow 0} \frac{2x-2}{2x+2}$$

THE LIMIT OF ...
AS x APPROACHES

TRY DIRECT
SUBSTITUTION

$$\frac{-2}{+2} = -1$$

$$\lim_{x \rightarrow 5} \frac{x^2-25}{x-5}$$

DIRECT SUBSTITUTION
NOT POSSIBLE, BECAUSE
 $\frac{0}{0}$ IS INDETERMINATE

$$\lim_{x \rightarrow 5} \frac{(x-5)(x+5)}{(x-5)} = x+5 = 10$$

PROVIDED
 $x \neq 5$

SPECIAL LIMIT

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

AS x INCREASES,
THE VALUE OF $\frac{1}{x}$
BECOMES SMALLER AND
SMALLER,
APPROACHING ZERO

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{500}, \frac{1}{10000}, \dots$$

ETC.

$$\lim_{x \rightarrow \infty} \frac{x^2-5x+1}{3x^2-2}$$

AS x APPROACHES
INFINITY.
NO DIRECT SUBSTITUTION
DIVIDE BOTH SIDES
BY x^2 (THE HIGHEST
POWER OF x IN THE
DENOMINATOR)

$$\frac{1 - \frac{5}{x} + \frac{1}{x^2}}{3 - \frac{2}{x^2}} = \frac{1}{3}$$

EXPONENT = LOGARITHM =

INDEX

122.

COMMON BASE 10

NATURAL OR NAPIERIAN

$$\log_{10} 100 = 2$$

BECAUSE $10^2 = 100$

$$\text{Base } e \div 2.718$$

$$\log_e 100 \div 4.6$$

 $\ln 100$ (SHORT FORM)

$$y = a^x$$

$$y = e^x$$

MEANS:

$$\log_a y = x$$

"GO AROUND THE BLOCK"

MEANS: $\log y$, BASE e IS x

$$\ln y = x$$

$$\log_2 64$$

CALCULATOR:

$$\frac{\ln 64}{\ln 2} = 6$$

$$\log_{\sqrt{2}} 32$$

CALCULATOR

$$\frac{\ln 32}{\ln \sqrt{2}} = 10$$

CALL IT x CALL IT x

$$\log_2 64 = x$$

$$\log_{\sqrt{2}} 32 = x$$

$$\text{MEANS } 2^x = 64 = 2^6$$

$$x = 6$$

$$2^{\frac{1}{2}x} = 32 = 2^5, x = 10$$

$$2^{x+1} = 32$$

$$5^{z-4} = 1$$

$$2^{x+1} = 2^5$$

$$x = 4$$

$$x+1 = \frac{\ln 32}{\ln 2}$$

$$x = 4$$

$$5^{z-4} = 5^0$$

$$z = 4$$

$$z-4 = \frac{0}{\ln 5} = 0$$

$$(\ln 1 = 0)$$

$$z = 4$$

LOGARITHM

123.

$$\text{LOG}_a xy = \text{LOG}_a x + \text{LOG}_a y$$

RULE 2

COMPARE INDEX RULE

$$\text{LET } x = a^k \text{ \& } y = a^m$$

THEN $xy = a^{k+m}$ MEANS

$$\text{LOG}_a xy = k+m = \text{LOG}_a x + \text{LOG}_a y$$

$$\text{LOG}_a \frac{x}{y} = \text{LOG}_a x - \text{LOG}_a y$$

RULE 3 = RULE 1

$$\text{LOG}_5 125$$

$$\text{LOG}_3 \sqrt{27}$$

$$\text{LOG}_a x^n = n \text{LOG}_a x$$

CALL IT X

$$\text{LOG}_5 125 = x$$

CALL IT X

$$3^x = 3^{3/2}$$

$$5^x = 5^3, x = 3$$

$$x = 1\frac{1}{2}$$

$$\ln 125 \div \ln 5 = 3$$

$$\ln \sqrt{27} \div \ln 3 = 1.5$$

$$3^{\text{LOG}_3 81}$$

$$2^{\text{LOG}_2 64}$$

$$e^{\ln 3}$$

$$e = e'$$

$$3^x = 81 = 3^4$$

$$x = 4$$

$$3^{\text{LOG}_3 81} = 3^4$$

$$= 81$$

SIMILARLY
VISUALISE
THE FORMAT:
ANSWER: 64

SIMILARLY

$$e^{\ln e} = e$$

$$\ln e = 1$$

$$e^{\ln x} = \ln 2$$

$$e^{\ln 3x} = 6$$

$$\text{LOG}_6 9 + \text{LOG}_6 4$$

$$\text{LOG}_2 32 - \text{LOG}_2 8$$

$$x = \ln 2$$

$$x \div .693$$

$$3x = 6$$

$$x = 2$$

$$\text{LOG}_6 36 = x$$

$$6^x = 6^2$$

$$x = 2$$

$$\ln 36 \div \ln 6 = 2$$

$$\text{LOG}_2 4 = .6$$

$$2 \text{LOG}_2 2$$

CHANGE OF BASE

$$y = \log_a x \quad (1)$$

$$x = a^y$$

$$\log_b x = \log_b a^y$$

$$= y \log_b a$$

SUB(1)

$$\log_b x = \log_a x \cdot \log_b a$$

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{VISUALISE}$$

AS A RESULT

$$\log_{10} x = \frac{\ln x}{\ln 10}$$

COMMON
NATURAL
NAPIERIAN
BASE e

$$\log_{10} 8.2 = \frac{\ln 8.2}{\ln 10}$$

.914 .914

$$y = \log_a x = \frac{\ln x}{\ln a \text{ (CONSTANT)}}$$

$$y' = \frac{1}{\ln a \cdot x}$$

$$y = \log_{10} x \rightarrow y' = \frac{1}{x \ln 10}$$

$$y = \log_{10} (4x-1) \rightarrow y' = \frac{4}{\ln 10 (4x-1)}$$

LOGARITHM

125

$$e^x = 600$$

$$e^{6.4x} = 600$$

$$34 = 24e^{.02t}$$

$$\text{LOG}_e 600 = x$$

THE UNDERSTANDING (ONCE)

e TO THE $x = 600$

THE DEFINITION OF LOG.

$$x \div 6.4$$

THE
ROUTINE

$$x = \frac{\ln 600}{6.4} \div 1$$

THE
ROUTINE

$$t = \frac{\ln(34 \div 24)}{.02}$$

$$t = 17.4$$

$$1 = 6e^{-.0069314t}$$

$$\text{LOG} 32 \div \text{LOG} 8$$

A ROUTINE STOPS
YOU FROM GETTING LOST IN
UNNECESSARY WORKING.

$$t = \frac{\ln(1 \div 6)}{-.0069314} \div 258$$

REGARDLESS OF BASE

$$\text{LOG} 2^5 \div \text{LOG} 2^3$$

$$5 \text{LOG} 2 \div 3 \text{LOG} 2 = \frac{5}{3}$$

$$e^{2.3026} = 10$$

CALCULATE
 $\ln 10, \ln \frac{1}{10}, \ln 100$

$$\ln 3 \div 1.099$$

$$\ln 2 \div .693$$

FIND
 $\ln 12$

BY DEFINITION

$$\ln 10 = 2.3026$$

$$\ln \frac{1}{10} = \ln 1 - \ln 10 = -2.3026$$

$$\ln 100 = 2 \ln 10 = 4.6052$$

THEORY: WITHOUT CALCULATOR

$$\ln 12 = \ln(3 \times 4) = \ln(3 \times 2^2)$$

$$= \ln 3 + 2 \ln 2 \div 2.485$$

BY
CHECK CALCULATOR

$$\text{LOG}_2 3$$

$$\text{LOG}_2 9$$

$$\text{LOG}_3 7$$

$$x^4 = 10000$$

CONVERSION

$$\frac{\ln 3}{\ln 2}$$

$$1.584963$$

$$2 \text{LOG}_2 3 \text{ OR}$$

$$\frac{\text{LOG} 9}{\text{LOG} 2}$$

(MEANS BASE 10)

$$\text{APPR. } 3.2$$

VISUALISE
CONVERSION

$$\ln 7 \leftarrow$$

$$\ln 3 \leftarrow$$

SAME POSITION

$$10000 x^{\frac{1}{4}}$$

(CASIO)

$$\text{APPR. } 17.8$$

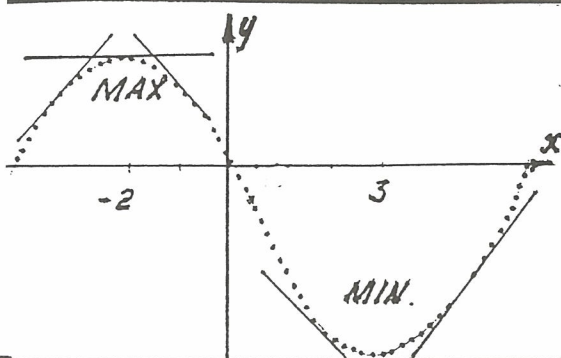
$$4 \text{LOG} x = 5$$

$$\text{LOG} x = 1.25$$

$$\text{INV. LOG } 17.8$$

MAXIMA AND MINIMA

126.



$$y = 2x^3 - 3x^2 - 36x + 4$$

$$y' = 6x^2 - 6x - 36$$

$$y'' = 12x - 6$$

ALWAYS
ANALYSE FROM L TO R

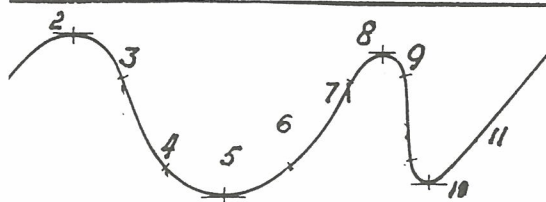
AS x
INCREASES

AT
STATIONARY POINTS, $y' = 0$ AND CHANGES SIGN
HORIZONTAL TANGENT

AT A MAXIMUM,
FROM POSITIVE TO NEGATIVE.
TANGENT FROM R TO L

AT A MINIMUM,
FROM NEGATIVE TO POSITIVE.
TANGENT FROM LEFT TO RIGHT

A CONTINUOUS DIFFERENTIABLE $f(x)$ IS INCREASING
WHEN $f'(x) > 0$, DECREASING WHEN $f'(x) < 0$



INCREASING AT 1, 6, 7 & 11
DECREASING AT 3, 4, 9
STATIONARY AT 2, 5, 8 & 10

$$y = 3x^4 + 4x^3 - 36x^2$$

$$y' = 12x^3 + 12x^2 - 72x$$

$$x = -3, 0, 2 \text{ S } x = -2, -1, 3 \text{ I } x = -4, -1 \text{ D}$$

READ $\rightarrow$ AS x INCREASES

x	-4	-3	-2	-1	0	1	2	3
y	-64	-108	-128	-37	0	-29	-64	27
y'	-288	0	96	72	0	-48	0	216

UNLESS $y' = y'' = 0$, CALCULATE y'' TO DETERMINE
THE NATURE OF THE STATIONARY POINTS.

AT A MAXIMUM, y' CHANGES FROM POSITIVE TO NEGATIVE.
SINCE y'' GIVES THE CHANGE OF y' ,

IT MUST BE < 0 (THINK GRAVITY, ROLLING DOWNHILL)

AT A MINIMUM, y'' MUST BE > 0 (THINK GROWING FORCE)

$$y' = x(x-2)(x+3) = 0$$

$$y'' = (3x^2 + 2x - 6)/2$$

HORIZONTAL TANGENTS
WHEN

$$x = 2$$

$$x = 0$$

$$x = -3$$

$$y'' > 0$$

$$y'' < 0$$

$$y'' > 0$$

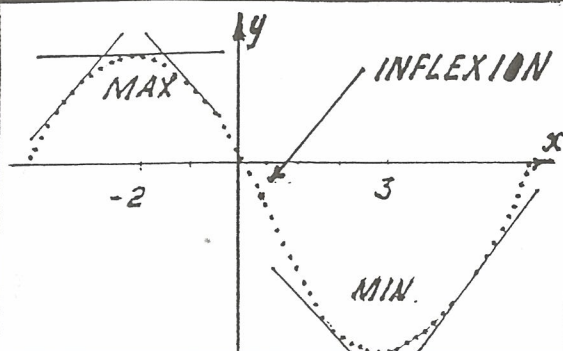
MIN. AT (2, -64)

MAX AT (0, 0)

MIN AT (-3, -109)

MAXIMA AND MINIMA

127



AT A POINT OF INFLEXION
THE TANGENT CHANGES SIDES.

THE $y'' = 0$ AND CHANGES SIGN.
TEST BEFORE AND AFTER.

$$y = 2x^3 - 3x^2 - 36x + 4$$

$$y' = 6x^2 - 6x - 36$$

$$y'' = 12x - 6 = 0$$

$$y'' < 0 \text{ AT } x = 0$$

$$y'' > 0 \text{ AT } x = 1$$

$$\text{INFLEXION } \left(\frac{1}{2}, -14\frac{1}{2}\right)$$

HORIZONTAL POINT OF INFLEXION

THE TANGENT IS HORIZONTAL, BUT IT IS NOT A T.P

$$y = x^5$$

$$y' = 5x^4$$

$$y'' = 20x^3$$

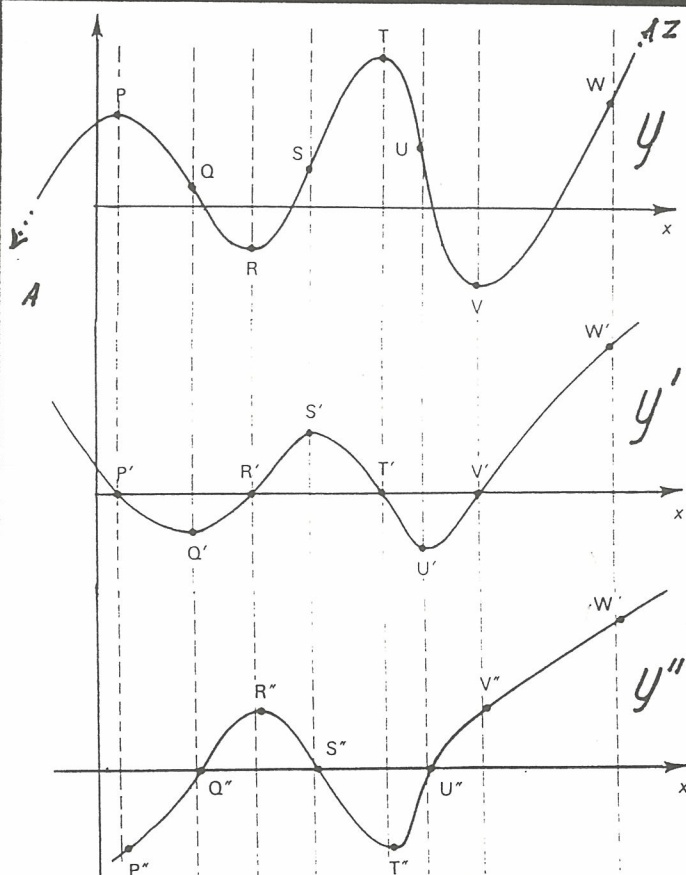
AT $(0,0)$ BOTH
 $y' \neq y''$ ARE ZERO

WHEN $x = -1, y'' < 0$

WHEN $x = 1, y'' > 0$

AT $x = \pm 1,$
 $y' = 5$

$(0,0)$ IS A
HOR. POINT OF INFL



CONCAVE DOWN

BETWEEN

$S \xi U, A \xi Q$

CONCAVE UP

BETWEEN $Q \xi S, U \xi Z$

CONCAVITY CHANGE

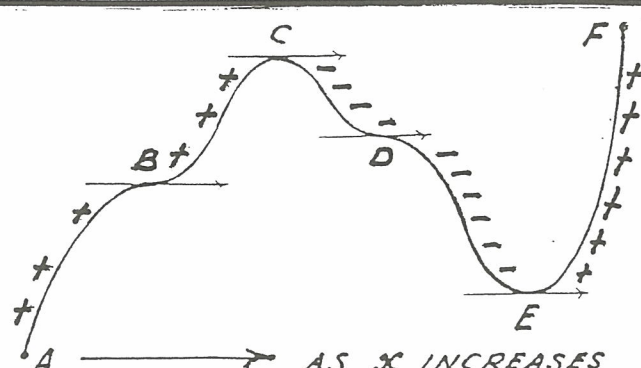
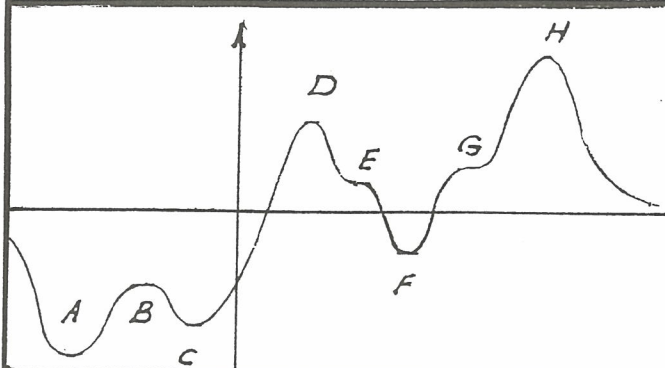
AT INFLEXIONS Q, S, U

$y' = 0$ AT $P, R, T \xi V$

$y'' = 0$ AT $Q, S \xi U$

MAXIMA AND MINIMA

128.



ABSOLUTE & LOCAL MINIMUM AT A

ABSOLUTE & LOCAL MAXIMUM AT H

LOCAL (RELATIVE) MIN. AT A, C, F
LOCAL (RELATIVE) MAX. AT B, D, H

COMPARE ONE TONE
MONOTONOUS

LOCAL MINIMUM AT E

LEAST VALUE AT A

LOCAL MAXIMUM AT C

GREATEST VALUE AT F

HORIZONTAL INFLEXION AT B, D
TANGENT CHANGES SIDES, NOT
SIGNS, HENCE y IS MONOTONIC
INCREASING ($y' > 0$) BETWEEN A & C.
MONOTONIC DECREASING (C → E)

$$y = 2x^2 - 8x + 7$$

$$y = (2x+1)^{-3}$$

$A > 0$. PARABOLA CONCAVE UP

$$y' = 4x - 8 = 0 \quad \text{MINIMUM}$$

$$y'' = 4 \quad \text{AT } (2, -1)$$

$$y' = -6(2x+1)^{-4}$$

ALWAYS < 0 ($x \neq -1/2$) $\lim = 0$

MONOTONIC INCREASING $x > 2$

MONOTONIC DECREASING, x REAL

$$y = x \ln x - x$$

$$y = \ln x + \frac{1}{x}$$

PROD. RULE

$$y' = \ln x = 0$$

$$x = 1$$

$$y'' = \frac{1}{x} > 0 \quad \text{AT } x = 1$$

MIN (1, -1)

NO INFLEXION: $y'' \neq 0$

$$y' = \frac{1}{x} - \frac{1}{x^2} = 0, x = 1 (x \neq 0)$$

$$y'' = -\frac{1}{x^2} + \frac{2}{x^3} = 1, \text{ AT } x = 1$$

MIN (1, 1)

$$y'' = 0, x = 2, \text{ TEST } x = 3, y'' < 0$$

INFLEXION AT $(2, \ln 2 + \frac{1}{2})$

MAXIMA AND MINIMA

129

2 NUMBERS > 0. SUM 20

MAXIMUM PRODUCT
 $y = x(20-x) = -\frac{1}{2}x^2 + 20x$
 CONCAVE DOWN
 MAX. WHEN $2x=20, x=10$
 EQUAL NUMBERS

RIGHT $\triangle$, SUM SIDES 12

MAXIMUM AREA
 $A = \frac{1}{2}x(12-x) = -\frac{1}{2}x^2 + 6x$
 $A' = -x + 6 = 0, x=6$
 $A'' < 0$. MAX. AREA = $18 u^2$

$$y = x^2 e^x$$

PROD. RULE INCL. F. OF F

$y' = 2x \cdot e^x + x^2 \cdot e^x = e^x(x^2 + 2x)$
 $y' = 0$ WHEN $x = 0 \text{ \& } -2 \rightarrow 0$
 $y'' = e^x(x^2 + 2x) + e^x(2x + 2)$
 $= e^x(x^2 + 4x + 2)$
 MIN. AT $(0, 0)$, MAX. AT $(-2, 4e^{-2})$

$$y = x / \ln x$$

PROD. RULE

$y' = \ln x + 1 = 0$
 $\ln x = -1$, MEANS $x = \frac{1}{e}$
 $y'' = \frac{1}{x} = e > 0$ MIN. $y = -\frac{1}{e}$

$$y = \frac{\ln x}{x}$$

$x^3 - 12x + 20$ $-3 \leq x \leq 5$

$y' = \frac{1 - \ln x}{x^2} = 0, x = e$

END VALUES 29 & 85

$\therefore y'' = -\frac{1}{x^3} \cdot x^2 - (1 - \ln x) \cdot 2x \div x^4$
 $(-3x + \ln x \cdot 2x) \div x^4 = \frac{-1}{e^3} < 0$
 SUB. $x = e$

$y' = 3x^2 - 12 = 0, x = \pm 2$

$y'' = 6x$, MIN. AT $(2, 4)$

MAX. AT (e, e^{-1})

MAX. AT $(-2, 36)$

WHEN $y'' = 0, 2x \ln x = 3x, x = e^{3/2}$
 TEST $x = e^2, y'' = \frac{1}{e^6} > 0$ $x = e, y'' < 0$

LEAST VALUE
 FOR THIS DOMAIN: 4

SIGN CHANGE
 INFLEXION AT $(e^{3/2}, \frac{3}{2}e^{-3/2})$

GREATEST VALUE: 85

MAXIMA

AND MINIMA

130.

SUM 2 RADII CONSTANT
AREA SUM LEAST WHEN $R_1 = R_2$

SPEED V km/hr.
COST PER HOUR $\$9000 + 10V^2$

FIND
 V
MIN
COST

1. THINK $y' = 0$
2. ELIMINATE ONE VARIABLE
3. DIFFERENTIATE
WITH RESPECT TO THE OTHER.

$$\text{TIME TO DO } D \text{ km} = \frac{D}{V}$$

$$R_1 + R_2 = C, R_2 = C - R_1$$

TOTAL COST

$$A = \pi(R_1^2 + R_2^2) = \pi(2R_1^2 - 2CR_1 + C^2)$$

$$(9000 + 10V^2) \frac{D}{V} =$$

$$10D \left(\frac{900}{V} + V \right)$$

$$A' = 2\pi(2R_1 - C) = 0$$

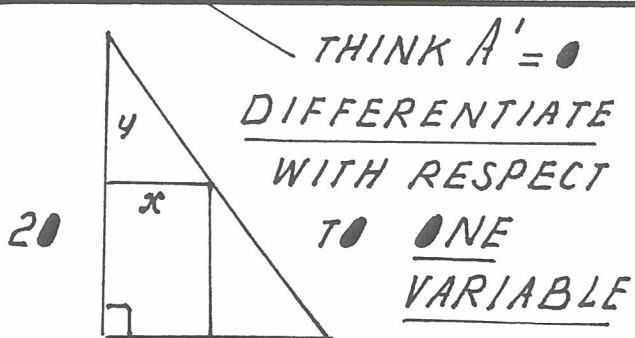
$$\text{MIN. WHEN } 1 - \frac{900}{V^2} = 0$$

$$2R_1 = C \therefore R_1 = R_2 \text{ q.e.d}$$

$$\text{WHEN } V = 30 \text{ km/hr.}$$

MAX. AREA RECTANGLE

BOX. NO LID. SQUARE BASE.
 $V = 500 U^3$. MINIMUM METAL.



THINK $A' = 0$
DIFFERENTIATE
WITH RESPECT
TO ONE
VARIABLE

$$\frac{x}{15} = \frac{y}{20}$$

$$y = \frac{4}{3}x \rightarrow A = x(20 - \frac{4}{3}x)$$

$$A' = 20 - \frac{8}{3}x = 0, x = 7\frac{1}{2}$$

$$A_{\text{MAX}} = 75 U^2$$

$$\text{BASE} = x^2, H = \frac{500}{x^2}$$

$$\text{AREA } x^2 + \frac{2000}{x} \quad (1)$$

$$A' = 2x - \frac{2000}{x^2} = 0$$

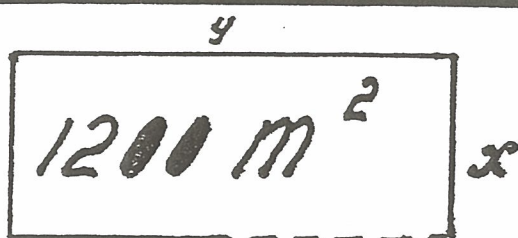
$$x^3 = 1000$$

$$x = 10, \text{ SUB (1)}$$

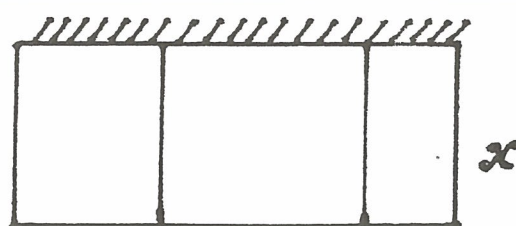
$$\text{MIN. METAL} = 300 U^2$$

MAXIMA AND MINIMA

131.



FENCE \$3 p.m
MINIMUM COST



MAXIMUM AREA
WITH 1 KM WIRE

$$P = 2x + \frac{1800}{x}$$

$$\text{COST} = 6x + \frac{5400}{x}$$

$$\text{COST}' = 6 - \frac{5400}{x^2} = 0$$

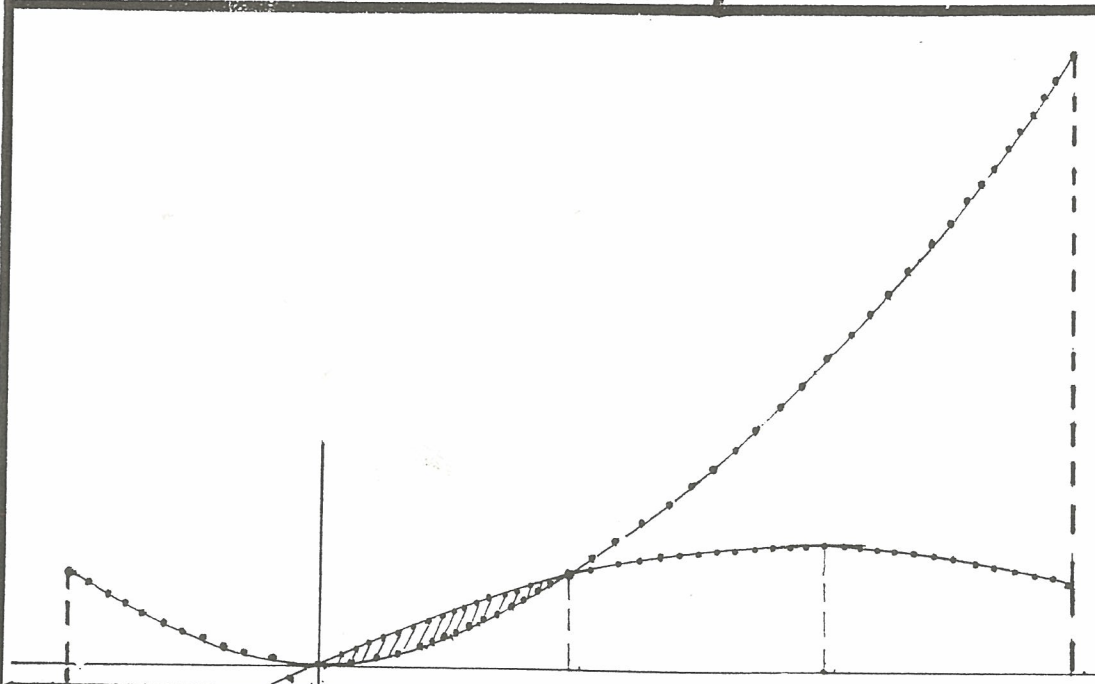
$$x = 30. \text{ MINIMUM COST } \$360$$

$$A = 1000x - 4x^2$$

$$A' = 1000 - 8x = 0$$

$$x = 125m$$

$$\text{MAXIMUM AREA } 62,500 m^2$$



$$y = 3x^2$$

$$y' = 6x = 0$$

$$y'' = 6$$

MIN. AT (0,0)

$$y = 4x - x^2$$

$$\text{MAX. AT } (2,4)$$

INTERSECTIONS: $3x^2 = 4x - x^2$
 $(0,0) \text{ \& } (1,3)$

DOMAIN $-1 \leq x \leq 3$ SUB. $\uparrow$

END VALUES $(-1, -5) \text{ \& } (3, 3)$

$(-1, 3) \text{ \& } (3, 27)$

$$A = \left[2x^2 - \frac{x^3}{3} - x^3 \right]' = \frac{2}{3}x^2$$

MAXIMA AND MINIMA

132.

BOX + LIDS FOR TOP,
SQUARE RECTANGULAR
SIDES & FRONT
.27m² MAX. VOLUME

$$y = 3x^2 + \frac{Q}{x^2}$$

TURNING POINT AT $x=3$

$$4x^2 + 5xy = .27$$

$$y = \frac{.27 - 4x^2}{5x}$$

$$V = x^2y = \frac{.27x - 4x^3}{5}$$

$$V' = \frac{.27 - 12x^2}{5} = 0$$

$$x = .15 \quad \text{MAX. } V = .0054 \text{ m}^3$$

$$y' = 6x - \frac{2Q}{x^3} = 0 \quad \text{WHEN } x=3$$

$$18 - \frac{2Q}{27} = 0$$

$$Q = 243$$

Box $2x \times x \times y$ cm
NO LID. SURFACE AREA
IS 150. SHOW $x \leq 5\sqrt{3}$
 x VALUE FOR V_{MAX} .

$$y = x(x-3)^2$$

POINT OF INFLEXION,
 x -INT. & TURNING POINTS

$$2x^2 + 6xy = 150$$

$$y = \frac{75 - x^2}{3x}$$

$$V = 2x^2y = \frac{150x - 2x^3}{3} \quad \text{?}$$

$$x \leq 5\sqrt{3}, \quad x \neq 0$$

$$V' = \frac{1}{3}(150 - 6x^2) = 0, \quad x = 5$$

$$V_{\text{MAX}} = 166\frac{2}{3} \text{ u}^3$$

x -INTERCEPTS AT $(0,0)$ & $(3,0)$

$$y = x^3 - 6x^2 + 9x$$

$$y' = 3x^2 - 12x + 9 = 0$$

$$(x-1)(x-3) = 0$$

$$y'' = 6x - 12$$

MAX. AT $(1,4)$

MIN. AT $(3,0)$

$$y'' = 6x - 12 = 0$$

& SIGN CHANGE

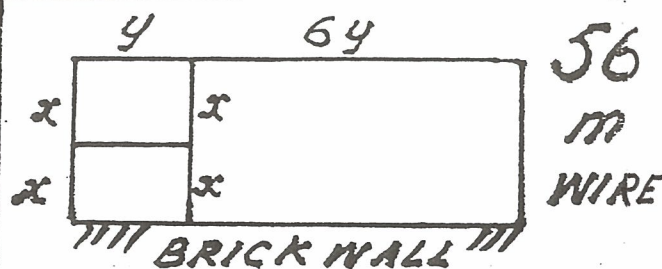
→ INFLEXION
AT $(2,2)$

MAXIMA AND MINIMA

133

$$y = x - 3/\ln x$$

$$1 \leq x \leq 7$$



$$y' = 1 - \frac{3}{x^2} = 0, x = 3$$

$$y'' = \frac{6}{x^3}, \text{ ALWAYS } > 0$$

REL. MIN. AT (3, 3 - 3/\ln 3)

END VALUES: 1 & 7

MAX. VALUE: 7 - 3/\ln 7 \approx 1.16

$$8y + 6x = 56$$

$$y = 7 - \frac{3}{4}x$$

$$A = 14x(7 - \frac{3}{4}x) = 98x - 10\frac{1}{2}x^2$$

$$A' = 98 - 21x = 0, x = \frac{4\frac{2}{3}}{1}m, 6x = 28$$

$$y = 3\frac{1}{2}m, 8y = 28$$

$$y' = x^2(2x-1)(x-1)$$

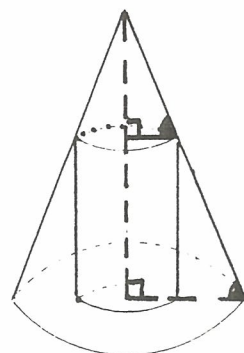
NATURE STATIONARY POINT AT $x=0$

$$y'' = 8x^3 - 9x^2 + 2x = 0$$

WHEN $x=0 \therefore$

TEST $x=-1: y' > 0$
 TEST $x=1/4: y' > 0$
 $y'' < 0$ & $y'' > 0$

HORIZONTAL POINT OF INFLEXION



SIMILAR TRIANGLES

MAX. VOLUME CYLINDER, $r \in h$

CONE $R6, H20$

$$\frac{h}{20} = \frac{6-r}{6}$$

$$h = \frac{10(6-r)}{3}$$

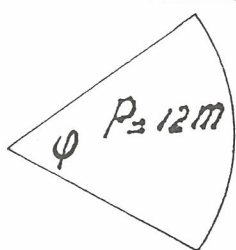
$$V = \frac{10\pi r^2(6-r)}{3} = 20\pi r^2 - \frac{10\pi r^3}{3}$$

$$V' = 40\pi r - 10\pi r^2 = 0$$

$$r = 4, h = 6\frac{2}{3} \text{ cm}$$

$r \neq 0$

$$\text{MAX } V = \frac{320\pi}{3} \text{ cm}^3$$



$$\phi R + 2R = 12$$

$$\phi = \frac{12-2R}{R}$$

$$A = \frac{\phi R^2}{2} = 6R - R^2$$

$$A' = 6 - 2R = 0, R = 3$$

$$A_{\text{MAX}} = 9\text{m}^2$$

USING TRIANGLE FORMULA!

PARABOLA

134.

$$x^2 + 2x - 3$$

$$x^2 + 2x - 3 = 0$$

QUADRATIC EXPRESSION

QUADRATIC POLYNOMIAL
many names
(SPECIFICALLY: TRINOMIAL)

QUADRATIC EQUATION

THE VALUES OF x
ARE CALLED
ZEROS OR ROOTS

FACTORISED

$$(x+3)(x-1)$$

$$(x+3)(x-1) = 0$$

$$x = -3$$

$$x = 1$$

THE GRAPH OF A PARABOLA

$$y = x^2 + 2x - 3$$

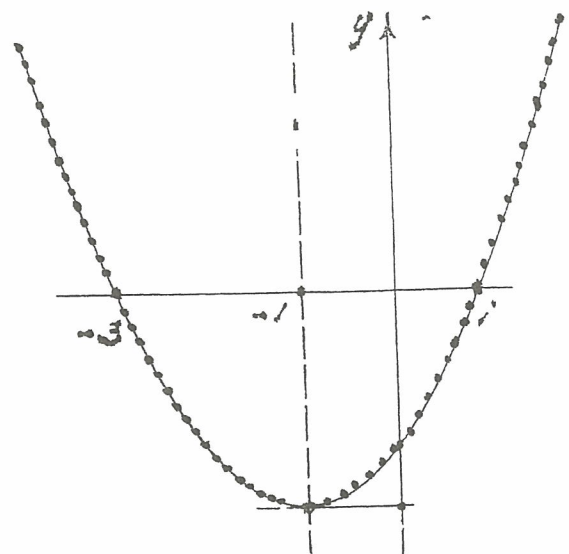
QUADRATIC FUNCTION

IF $y = 0$, THE ROOTS OR
ZEROS ARE THE
 x -INTERCEPTS

AXIS OF SYMMETRY $x = -1$

SUBSTITUTE IN $y = x^2 + 2x - 3$

VERTEX $(-1, -4)$ y -INTERCEPT $(0, -3)$



PARABOLA

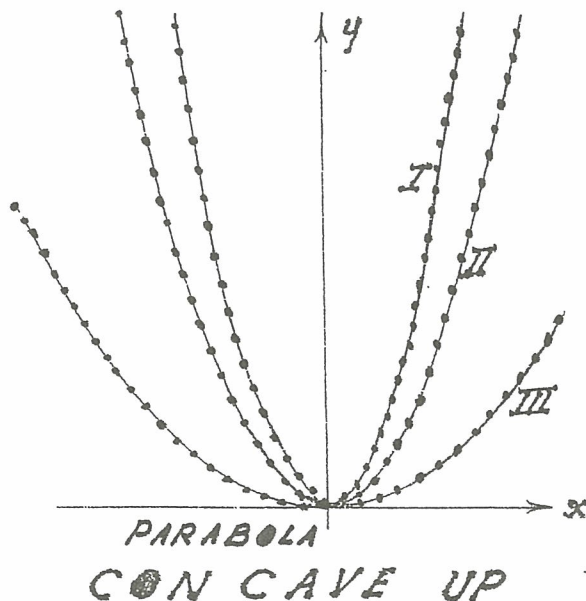
135.

$$y = 2x^2 \quad \text{I}$$

$$y = x^2 \quad \text{II}$$

$$y = \frac{1}{2}x^2 \quad \text{III}$$

THE COEFFICIENT OF x^2
 > 0



THE VALUE OF THE
QUADRATIC EXPRESSIONS
IS ZERO WHEN $x=0$.
FOR ALL OTHER VALUES OF x
THE EXPRESSIONS ARE
POSITIVE

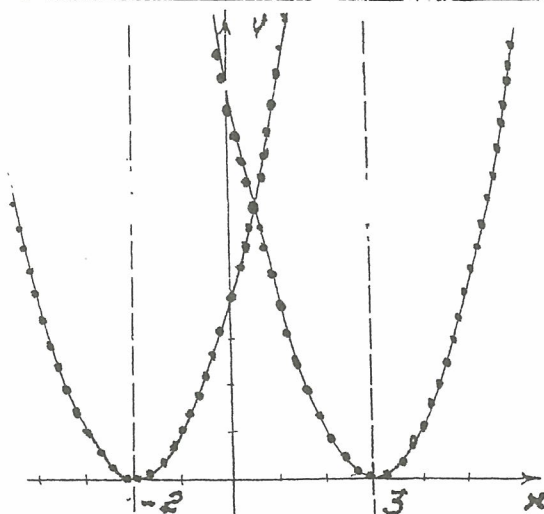
THE COEFFICIENT OF x^2
INFLUENCES THE SHAPE

EQUAL ROOTS
- REAL -

$$y = (x+2)^2 \quad \text{left}$$

$$y = (x-3)^2 \quad \text{right}$$

POSITIVE COEFFICIENTS OF x^2
PARABOLAS CONCAVE UP



$y = (x+2)^2$ EQUAL ROOTS
 $x = -2, y = 0$

EQUATION
AXIS OF SYMMETRY $x = -2$

$y = (x-3)^2$ EQUAL ROOTS
 $x = 3, y = 0$

EQUATION
AXIS OF SYMMETRY
 $x = 3$

PARABOLA

136.

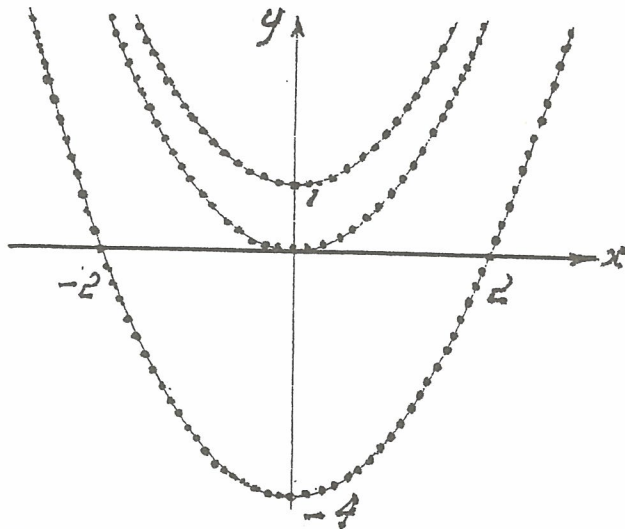
$$y = x^2 + 1$$

COEFFICIENT
POSITIVE ONE

$$y = x^2$$

$$y = x^2 - 4$$

$$= (x-2)(x+2)$$



$$y = x^2 + 1$$

NO REAL ROOTS.

POSITIVE DEFINITE
FUNCTION

$$y = x^2$$

EQUAL ROOTS.

ZERO OR POSITIVE
FUNCTION

$$y = x^2 - 4$$

TWO ROOTS.

POSITIVE COEFFICIENTS OF x^2
CONCAVE UP

INDEFINITE FUNCTION
ZERO, POSITIVE OR NEGATIVE

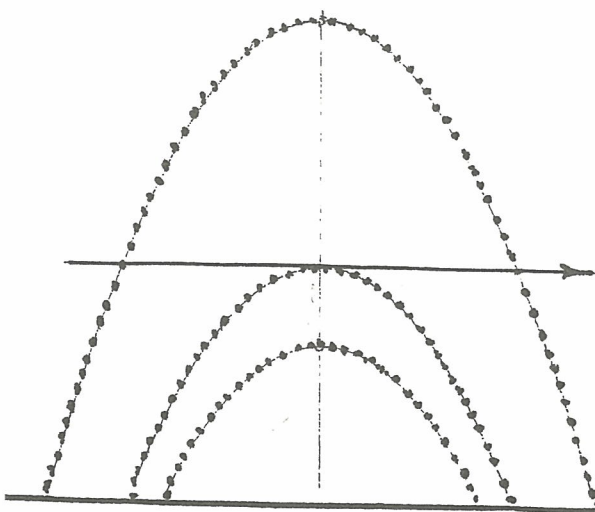
$$y = 4 - x^2$$

$$(2-x)(2+x)$$

$$y = -x^2$$

COEFFICIENT -1

$$y = -x^2 - 1$$



$$y = 4 - x^2$$

TWO ROOTS.

INDEFINITE FUNCTION

$$y = -x^2$$

EQUAL ROOTS.

ZERO OR NEGATIVE
FUNCTION

$$y = -x^2 - 1$$

NO REAL ROOTS.

NEGATIVE COEFFICIENT OF x^2
CONCAVE DOWN

NEGATIVE
DEFINITE FUNCTION

PARABOLA

137.

THE QUADRATIC FORMULA

FOR THE GENERAL QUADRATIC EQUATION $ax^2+bx+c=0$

x -INTERCEPTS
OF $y = ax^2+bx+c$ ($y=0$)

DISCRIMINANT Δ

$$b^2-4ac$$

REAL ROOTS

WHEN $\Delta=0$ (EQUAL ROOTS)

WHEN $\Delta>0$ (2 ROOTS)

NO REAL ROOTS

WHEN $\Delta<0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$2a$$

EQUATION AXIS
OF SYMMETRY

$$x = \frac{-b}{2a}$$

$$y = 2x^2 - 3x - 7$$

$$y = 3x^2 - 10x + 8$$

PARABOLA CONCAVE UP

PARABOLA CONCAVE UP

$$\Delta = 9+56=65$$

2 ROOTS

$$\Delta = 100-96=4$$

2 ROOTS

EQUATION AXIS

$$x = \frac{3}{4}$$

EQUATION AXIS

$$x = 1\frac{2}{3}$$

MINIMUM VALUE

$$y = -8\frac{1}{8}$$

MINIMUM VALUE

$$y = -\frac{1}{3}$$

VERTEX

$$\left(\frac{3}{4}, -8\frac{1}{8}\right)$$

VERTEX

$$\left(1\frac{2}{3}, -\frac{1}{3}\right)$$

$$y=0 \text{ FOR } x = \frac{3 \pm \sqrt{65}}{4}$$

$y < 0$ FOR x IN BETWEEN ROOTS

$y > 0$ FOR x OUTSIDE ROOTS

$$y=0 \text{ FOR } x = 1\frac{1}{3} \text{ OR } x = 2$$

$y < 0$ FOR $1\frac{1}{3} < x < 2$

$y > 0$ FOR $x > 2$ OR $x < 1\frac{1}{3}$

PARABOLA

138.

VALUES OF x DOMAIN AND RANGE VALUES OF y

$$y = \sqrt{4-x}$$

$$y = \sqrt{x-5} + \frac{1}{2x-15}$$

DOMAIN

$$4-x \geq 0$$

$$x \leq 4$$

DOMAIN

$$x \geq 5$$

EXCEPT

$$7\frac{1}{2}$$

RANGE

$$y \geq 0$$

RANGE

$$y \geq -\frac{1}{5}$$

$$\lim_{x \rightarrow \infty} \frac{1}{2x-15} = 0 \text{ AS } x \rightarrow \infty$$

$$y = 2x^2 - kx + 3$$

SYMMETRICAL ABOUT $x = -3$

$$\alpha + \beta = 3 \quad \alpha\beta = -2$$

THE FORMULA FOR THE AXIS OF SYMMETRY

IS
$$\frac{-b}{2a}$$

VISUALISE ONLY

α AND β ARE

THE ZEROS, ROOTS OR x -INTERCEPTS OF THE PARABOLA

$$\frac{k}{4} = -3$$

SUB. MENTALLY

$$k = -12$$

$$y = (x-\alpha)(x-\beta) \\ = x^2 - (\alpha+\beta)x + \alpha\beta$$

$$y = 2x^2 + 12x + 3$$

$$y = x^2 - 3x - 2$$

PARABOLA

139.

$$y = ax^2 + bx + c$$

through $(0,1)$, $(1,0)$ & $(2,3)$ CHECK $P(-1,6)$

SUBSTITUTE $x=0, y=1$ $\therefore c=1$

SUBSTITUTE $x=1, y=0$
 $c=1$ $0 = a + b + 1$
 $0 = 2a + 2b + 2$

SUBSTITUTE $x=2, y=3$
 $c=1$ $3 = 4a + 2b + 1$
 $0 = 2a + 2b + 2$ ←

 $3 = 2a - 1$
 $a = 2, b = -3$

SUBSTITUTE $x=-1, y=6$
 $6 = 2 + 3 + 1$, TRUE $\therefore P$ ON PARABOLA

$$y = 2x^2 - 3x + 1$$

$$y = x^2 - (k+2)x + 2k + 4$$

CALCULATE THE VALUES OF k FOR WHICH THE ROOTS ARE
EQUAL, REAL, NOT REAL, REAL AND DIFFERENT

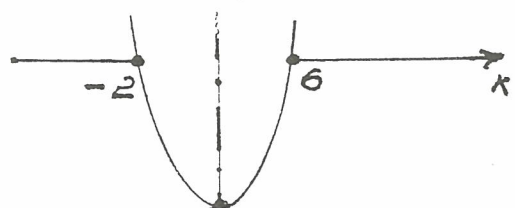
Δ

$$k^2 + 4k + 4 - 8k - 16$$

$$k^2 - 4k - 12$$

$$(k-6)(k+2)$$

$$\Delta = (k-6)(k+2)$$



EQUAL ROOTS	$k=6$ & $k=-2$	$\Delta = 0$ WHEN $k=6$ & $k=-2$
REAL ROOTS	$k \leq -2$ & $k \geq 6$	$\Delta \geq 0$ WHEN $k \leq -2$ & $k \geq 6$
NO REAL ROOTS	$-2 < k < 6$	$\Delta < 0$ WHEN $-2 < k < 6$
REAL & DIFFERENT ROOTS	$k < -2$ & $k > 6$	

PARABOLA

140.

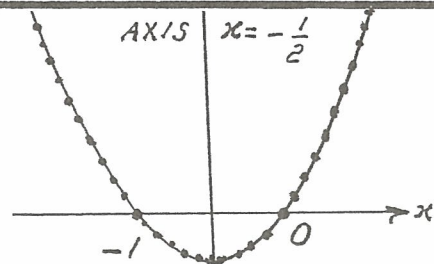
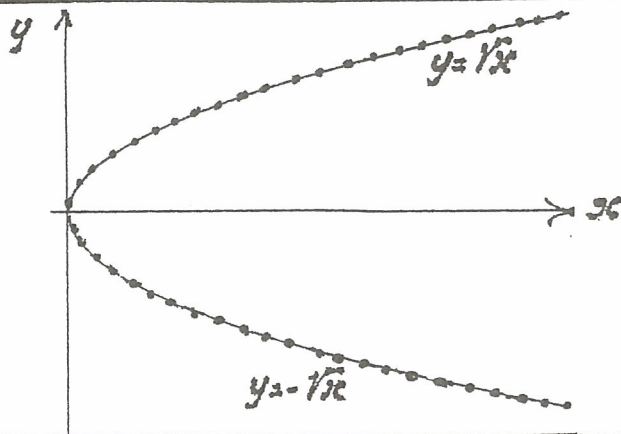
THE INVERSE OF $y = x^2$

$$x = y^2$$

$$y = \pm \sqrt{x}$$

$$x^2 + x \geq 0$$

QUADRATIC INEQUALITY



$$y = x^2 + x = x(x+1)$$

ZERO FOR $x = 0$ & $x = -1$

POSITIVE FOR

$$x < -1 \text{ \& } x > 0$$

SOLUTION (TRUTH) SET

$$\{x: x \leq -1\} \cup \{x: x \geq 0\}$$

$x = y^2$ IS A RELATION

THE BRANCHES ARE FUNCTIONS.

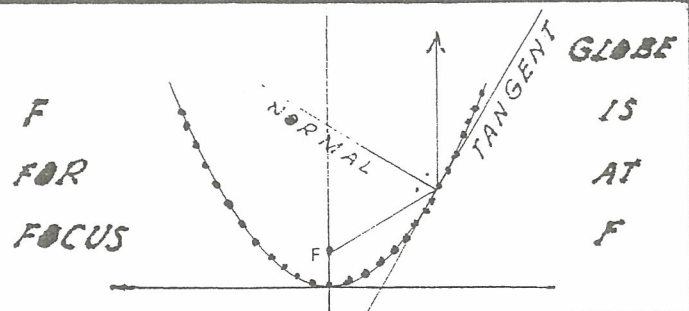
$\sqrt{x}$ IS THE REFLECTION OF $-\sqrt{x}$ IN THE x -AXIS

CURVE OPENS TO THE RIGHT

PRACTICAL APPLICATION.
12 CM WIRE SHAPED INTO
RECTANGLE. $A = 6 \text{ cm}^2$

PRACTICAL APPLICATION.
HEADLIGHTS
SEARCH LIGHTS

$$\begin{array}{|c|} \hline 6 \text{ cm}^2 \\ \hline \frac{6}{x} \\ \hline \end{array} \quad x$$



$$2x + \frac{12}{x} = 12$$

$$2x^2 - 12x + 12 = 0$$

$$x^2 - 6x + 6 = 0$$

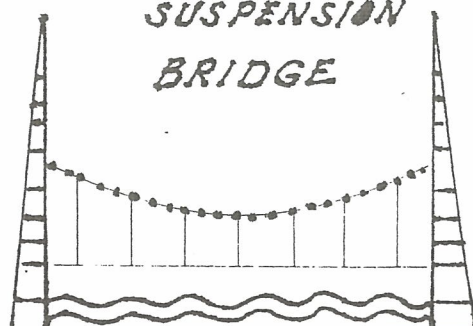
$$x = 3 \pm \sqrt{3}$$

$$y = 3 \mp \sqrt{3}$$

SIDES 4.73
+ 1.27

$\frac{6}{\text{cm}}$
 $P = 12$; TRUE

SUSPENSION
BRIDGE

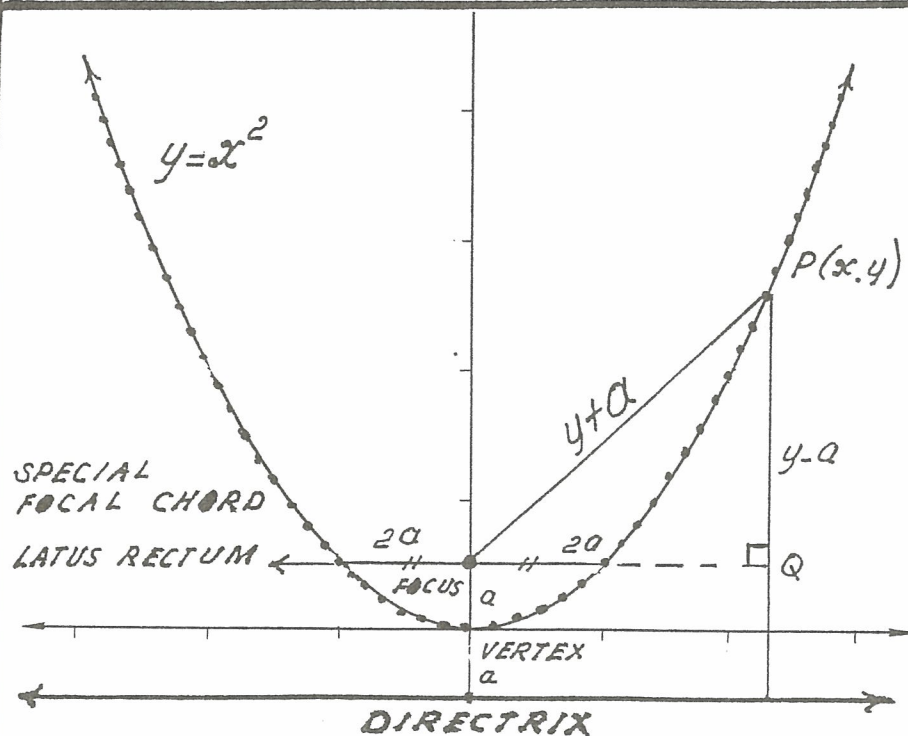


PARABOLA DEGREE 2.

141.

AS THE

LOCUS (PATH) OF A POINT MOVING SO THAT ITS DISTANCE FROM **FOCUS** EQUALS DISTANCE FROM **DIRECTRIX**
GIVING DIRECTION & SHAPE



TRIANGLE FPQ

$$x^2 = (y+a)^2 - (y-a)^2$$

$$x^2 = 4ay$$

STANDARD
PARABOLA

FOCUS (0, a)

VERTEX (0, 0)

FOR $y = x^2$

$$a = \frac{1}{4}$$

$$y = 2x^2$$

FOCUS & DIRECTRIX

$$y = -\frac{1}{2}x^2$$

FOCUS & DIRECTRIX

CONCAVE UP

CONCAVE DOWN

STANDARD PARABOLA
VERTEX THE ORIGIN

STANDARD PARABOLA
VERTEX THE ORIGIN

$$x^2 = \frac{1}{2}y = 4ay$$

$$x^2 = -2y = 4ay$$

$$a = \frac{1}{8}$$

FOCUS (0, $\frac{1}{8}$)

DIRECTRIX
 $y = -\frac{1}{8}$

$$a = -\frac{1}{2}$$

FOCUS (0, $-\frac{1}{2}$)

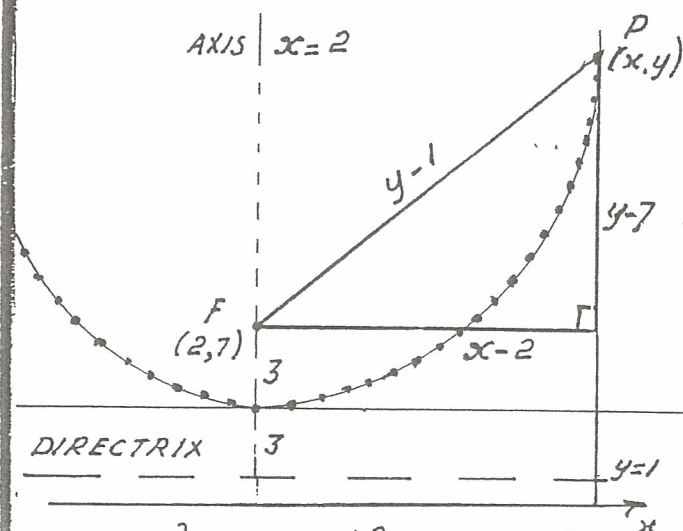
DIRECTRIX $y = \frac{1}{2}$

PARABOLA

142.

VERTEX (2,4) FOCUS
(2,7)

THE INVERSE OF
 $y = x^2$



$$x = y^2$$

STANDARD PARABOLA
(SIDEWAYS)

$$y^2 = 4ax$$

$$(y-1)^2 - (y-7)^2 = (x-2)^2$$

$$12y = x^2 - 4x + 52$$

MINIMUM VALUE OF $y = x^2 - 2x - 8$
CONCAVE UP

AXIS OF SYMMETRY
FORMULA

COMPLETING
THE SQUARE

m TANGENT = $y' = 0$

$$x = 1$$

$$y = (x-1)^2 - 9$$

SQUARES $\gg 0$

$$y' = 2x - 2 = 0$$

$$x = 1$$

$$y_{\text{MIN.}} = -9$$

$$y_{\text{MIN.}} = -9$$

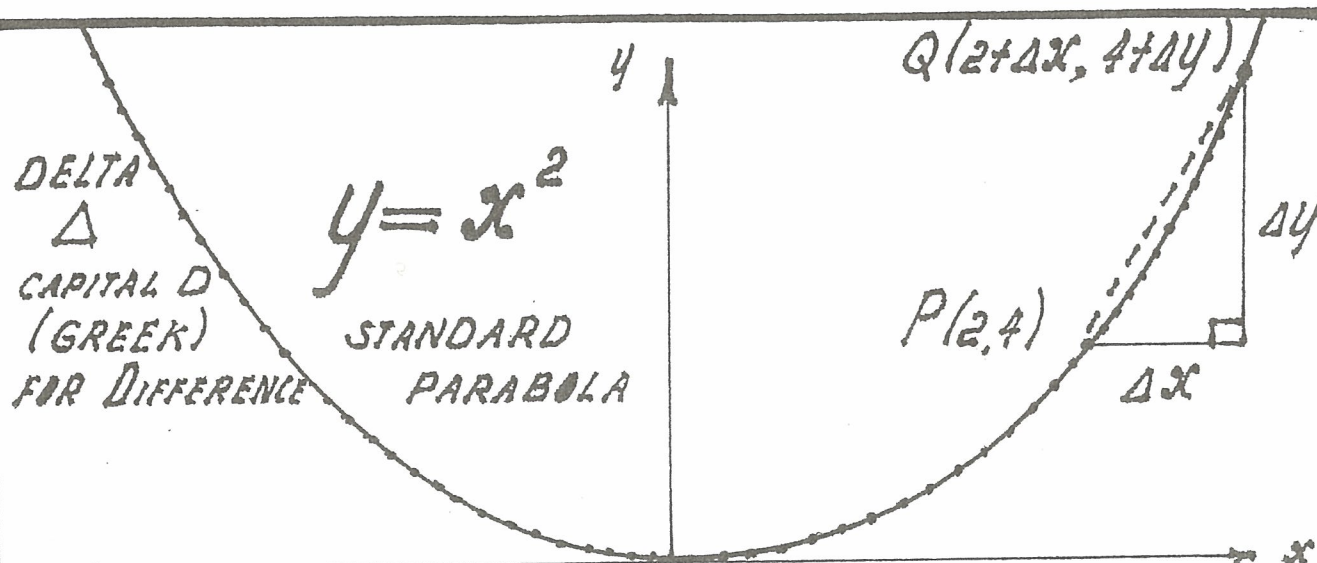
$$y_{\text{MIN.}} = -9$$

PARABOLA

143.

FROM FIRST PRINCIPLES

MEANS: PROVING THE ROUTINE ('FROM SCRATCH')



AVERAGE GRADIENT $PQ = \frac{\Delta y}{\Delta x}$

Q LIES ON THE PARABOLA, SO ITS COORDINATES SATISFY $y = x^2$

$$4 + \Delta y = 4 + 4\Delta x + \Delta x^2$$

$$\frac{\Delta y}{\Delta x} = 4 + \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{d}{dx} = y' = m_{\text{TANGENT AT P}} = 4$$

FROM SECANT TO TANGENT

ROUTINE

$$y' = 2x$$

= 4 AT P

PARABOLA

144.

GRADIENT OF $y = x^2$ AT ANY POINT P (FROM FIRST PRINCIPLES)

USE $P(x, y)$ & $Q(x + \Delta x, y + \Delta y)$

LEARN PROCEDURES, NOT DETAILS

SMALL INCREMENTS

SUB Q IN $y = x^2$

1. y
3.

$$y + \Delta y = (x + \Delta x)^2$$

2. SUBTRACT $y = x^2$

AND DIVIDE BY Δx

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x \quad (\Delta x \neq 0)$$

AS Δx APPROACHES ZERO, $y' = 2x$

WHICH CONFIRMS THE ROUTINE.

AT $(-3, 9)$, THE GRADIENT OF THE TANGENT $= y' = -6$
LEANING LEFT

GRADIENT FUNCTION

$$y = 4x^2 - 2x + 6$$

FIRST PRINCIPLES

AT Q¹, $(y + \Delta y) = 4(x + \Delta x)^2 - 2(x + \Delta x) + 6$

$$\frac{\Delta y}{\Delta x} = 8x + 4\Delta x - 2$$

AS Δx APPROACHES 0, $y' = 8x - 2$ (ROUTINE) AS

2. SUBTRACT
 $y = 4x^2 - 2x + 6$

3. AND DIVIDE BY
 $\Delta x \neq 0$

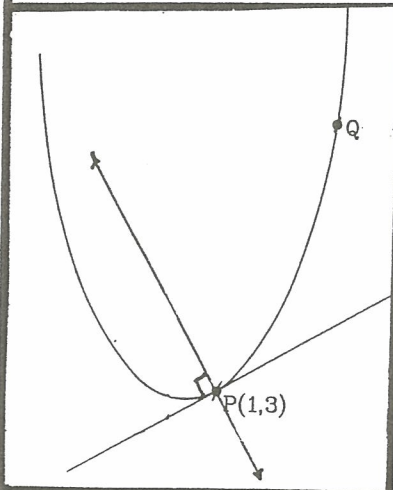
PARABOLA

145.

$$y = 2x^2 + 1$$

FIND GRADIENT FUNCTION

EQUATIONS OF THE TANGENT & NORMAL AT $P(1,3)$ ON $y = 2x^2 + 1$



$$Q(4,33)$$

$$y' = 4x$$

$$\text{AT } Q, y' = 16$$

$$\text{AT } P, y' = 4$$

$y' = 4x$ IS THE GRADIENT FUNCTION,
THE EQUATION OF TANGENT
AT ANY POINT $P(x,y)$
ON THE CURVE

m TANGENT AT P IS 4

$$3 = 4 + b$$

EQUATION TANGENT

$$y = 4x - 1$$

m NORMAL AT $P = -1/4$

$$3 = -1/4 + b$$

EQUATION NORMAL

$$4y = -x + 13$$

$$y = x^2 \text{ TANGENT AT } P(3,9)$$

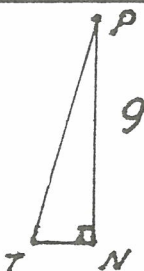
CUTS x -AXIS AT T . $PN \perp x$ -AXIS
FIND TN

$$y = \frac{1}{4}x^2 \text{ EQUATION NORMAL AT } (4,4)$$

FIND 2ND. INTERSECTION
AND CHORD LENGTH

$$y' = 2x$$

$$\text{AT } P, y' = 6$$



$$y' = \frac{1}{2}x. \text{ AT } (4,4), y' = 2$$

$$m \text{ NORMAL} = -1/2 \quad 4 = -2 + b$$

EQUATION NORMAL

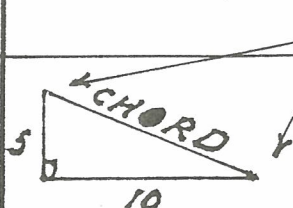
$$y = -1/2x + 6$$

$$y' = m \text{ TANGENT} = \frac{9}{TN}$$

$$TN = \frac{9}{6} = 1\frac{1}{2} \text{ UNITS}$$

AT
INTERSECTION $-1/2x + 6 = 1/4x^2$
 $(x+6)(x-4) = 0$
INTERSECTIONS $(4,4)$ & $(-6,9)$

6 INV TAN
ANGLE $PTN \div 81^\circ$



CHORD LENGTH
 $5\sqrt{5}$ UNITS

PARABOLA

146.

$$y = ax^2 + bx + c$$

$A(-1,4) \quad B(0,7) \quad C(1,8)$
ARE ON THE PARABOLA
CHECK $D(3,4)$

$$x^2 + (k-3)x + k = 0$$

VALUES k REAL ROOTS
VALUES k EXPRESSION
ALL VALUES x

$$c = 7$$

$$\begin{cases} 4 = a - b + 7 \\ 8 = a + b + 7 \end{cases} \quad \begin{cases} a = -1 \\ b = 2 \end{cases}$$

$$y = -x^2 + 2x + 7$$

SUBSTITUTE Δ CO-ORDINATES
 D IS ON THE PARABOLA

$$(k-3)^2 - 4k \geq 0$$

1 ROUTINE:
DISJOINT
SETS

CONFIRM:
TEST $k=3$,
FALSE

$$(k-9)(k-1) \geq 0$$

EQUAL ROOTS (1) $k=1, k=9$
2 ROOTS $k < 1, k > 9$
 $\therefore$ REAL ROOTS $k \leq 1, k \geq 9$

$$1 < k < 9, \Delta < 0$$

$\therefore$ NO REAL ROOTS

SINCE $Q > 1$, EXPRESSION > 1
FOR ALL VALUES OF x (POS. DEF)

$$y = 3x^2 + bx + 10$$

SYMMETRICAL ABOUT
THE LINE $x = 4$

$P(6,9) \quad Q(-2,1)$ ON $x^2 = 4y$
T INTERSECTION TANGENTS
MIDPOINT PQ
SHOW CURVE BISECTS TM

$$\frac{-b}{6} = 4$$

$$b = -24$$

EFFICIENCY

THE
FORMULA
IS IN YOUR
HEAD;
NO NEED
TO WRITE IT

$$y = \frac{x^2}{4}, \quad y' = \frac{x}{2}$$

$$\left. \begin{aligned} y'(P) &= 3 \\ 9 &= 18 + b \end{aligned} \right\} y(P) = 3x - 9$$

$$\left. \begin{aligned} y'(Q) &= -1 \\ 1 &= 2 + b \end{aligned} \right\} y(Q) = -x - 1$$

$$T(2, -3)$$

Axis & Vertex $y = 5x^2 + 10x + 2$

MENTALLY $(-1, -3)$

MIDPOINT PQ $(2, 5)$
MIDPOINT TM $(2, 1)$

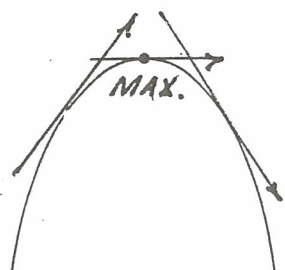
SUB. $y = \frac{x^2}{4}$, TRUE

PARABOLA

147.

MAXIMUM, MINIMUM TURNING POINTS

THE 2ND. DERIVATIVE (INVISIBLE) SHOWS THE CHANGE OF THE 1ST.



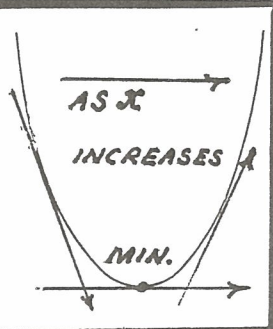
$$y = -x^2 + 25$$

$$y' = -2x$$

$$y'' = -2$$

AS x INCREASES $\rightarrow$

THE NEGATIVE y'' CAUSES THE y' TO CHANGE FROM POSITIVE TO NEGATIVE THROUGH THE MAX.T.P (THINK: DOWN HILL)



$$y = x^2 - 4x + 2$$

$$y' = 2x - 4$$

$$y'' = 2$$

THE POSITIVE y'' CAUSES THE TANGENT TO CHANGE FROM LEFT ($m = y' < 0$) TO RIGHT ($m = y' > 0$) THROUGH THE MIN.T.P. (THINK: GROWING FORCE)

MAX. HEIGHT STONE THROWN UPWARDS: $h = -5t^2 + 40t$

AT MAXIMUM

$$y' = -10t + 40 = 0$$

HORIZONTAL
TANGENT

$$t = 4$$

MAX. HEIGHT, AFTER 4 SEC.,
80 m

MINIMUM PERIMETER RECTANGLE: $A = 100 \text{ cm}^2$

SIDES x AND $\frac{100}{x}$

$$P = 2x + 200x^{-1}$$

$$P' = 2 - 200x^{-2} = 0$$

WHEN $x = 10$

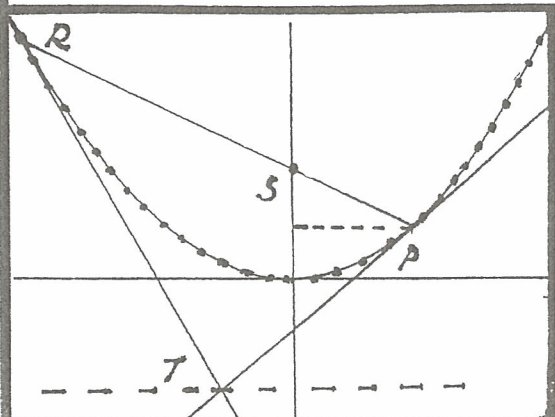
THE RECTANGLE IS ACTUALLY A SQUARE: $P = 40 \text{ cm}$.

PARABOLA

148.

$$16y = x^2$$

FOCUS S DIRECTRIX



$$y = \frac{x^2}{16}$$

$$y' = \frac{x}{8}$$

EQUATION TAN. $P(4,1)$

EQUATION TAN. $R(x,y)$

INTERSECTION T

ON DIRECTRIX

$$y'(P) = \frac{1}{2} \left\{ \begin{array}{l} y(P) = \frac{1}{2}x - 1 \\ 1 = 2 + b \end{array} \right.$$

$$m(PS) = -\frac{3}{4} \left\{ \begin{array}{l} y(PSR) = -\frac{3}{4}x + 4 \\ b = 4 \\ y = \frac{x^2}{16} \end{array} \right.$$

INTERSECTIONS

$$\left\{ \begin{array}{l} x^2 + 12x - 64 = 0 \\ (x+16)(x-4) = 0 \end{array} \right. \left\{ \begin{array}{l} P(4,1) \\ R(-16,16) \end{array} \right.$$

$$y'(R) = -2 \left\{ \begin{array}{l} y(R) = -2x - 16 \\ 16 = 32 + b \\ y(P) = \frac{1}{2}x - 1 \end{array} \right.$$

$T(-6,-4)$, WHICH IS ON THE DIRECTRIX

$S(0,4)$

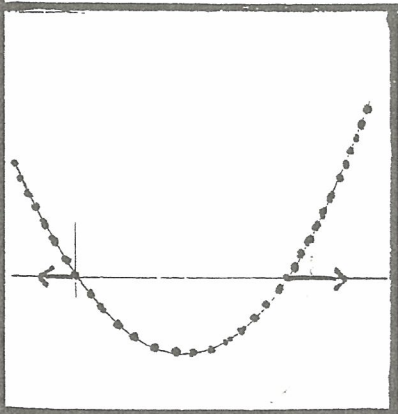
EQUATION DIRECTRIX $y = -4$

$$2x^2 \geq x$$

$$y = ax^2 + bx + c$$

THROUGH $(0,0)$

VERTEX $(2,1)$



$$2x^2 - x \geq 0$$

$$x(2x-1) \geq 0$$

$$\text{SUB. } (0,0) \quad c = 0$$

$$\text{SUB. } (2,1) \quad 4a + 2b = 1$$

$$\left\{ \begin{array}{l} \frac{-b}{2a} = 2 \\ b = -4a \end{array} \right.$$

$$a = -\frac{1}{4} \quad b = 1$$

$$y = -\frac{1}{4}x^2 + x$$

ZEROS $x = 0$ $x = \frac{1}{2}$

FOR $x \leq 0$ & $x \geq \frac{1}{2}$

PARABOLA ON OR ABOVE
X-AXIS (≥ 0)

PARABOLA

149.

$$Q(x) = ax^2 + bx + c$$

$$Q(0) = 4, Q(1) = 23, Q(-1) = 1$$

$$Q(x) = 0, \text{ NO SOLUTIONS}$$

$$y = x^2 - 4x + 6$$

AXIS & VERTEX

FOCAL LENGTH. FOCUS

$$Q(0) = 4, c = 4$$

$$\text{EQUATION AXIS } x = 2$$

$$Q(1) = 23, a + b + 4 = 23$$

$$a + b = 19$$

$$\text{VERTEX } (2, 2)$$

$$Q(-1) = 1, a - b + 4 = 1$$

$$a - b = -3$$

$$a = 8, b = 11$$

REWRITE & COMPLETE THE SQUARE

$$x^2 - 4x = y - 6$$

COMPARE
CENTRE CIRCLE
& ITS EQUATION

$$(x - 2)^2 = 1(y - 2)$$

$$4a = 1, a = \frac{1}{4} = \text{F.L.}$$

$$\text{FOCUS } (2, 2\frac{1}{4}), \text{ DIRECTRIX } y = 1\frac{3}{4}$$

$$\Delta = 121 - 128 < 0$$

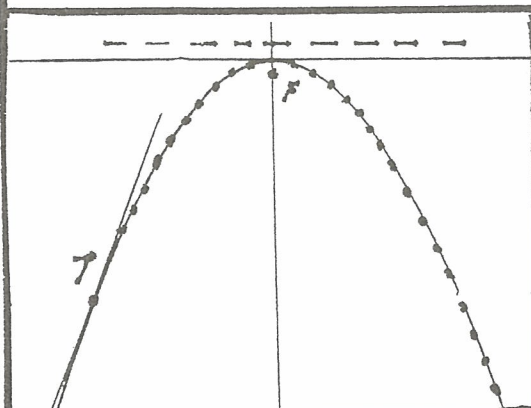
$a > 0$ NO REAL SOLUTIONS
PARABOLA POS. DEF. p.e.d.

$$y_1 = Bx^2, \text{ TANGENT } y_2 = 20x + 20$$

$$\text{FOCUS } (2, 5) \text{ DIRECTRIX } y = -3$$

Focus & DIRECTRIX

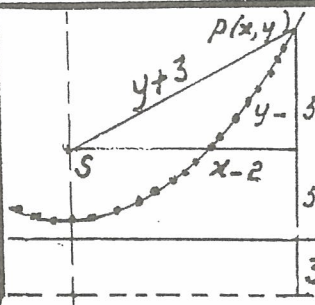
VERTEX & EQUATION
FOCAL LENGTH



AT THE
INTER-
SECTION

$$y_1 = y_2$$

1 REAL
ROOT $\Delta = 0$



$$\text{VERTEX } (2, 1)$$

$$\text{FOCAL LENGTH } 4$$

EQUATION

$$(x - 2)^2 = 16(y - 1)$$

$$Bx^2 - 20x - 20 = 0$$

$$-5x^2 - 20x - 20 = 0$$

$$400 + 80B = 0$$

$$(x + 2)^2 = 0 \therefore T(-2, -20)$$

$$x^2 = -\frac{1}{5}y \therefore a = -\frac{1}{20}$$

$$\text{FOCUS } (0, -\frac{1}{20}), \text{ DIRECTRIX } y = \frac{1}{20}$$

LONGER METHOD

$$(y + 3)^2 - (y - 5)^2 = (x - 2)^2$$

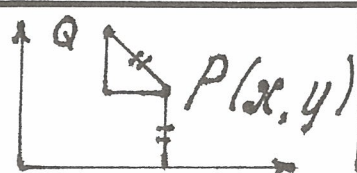
$$16y = x^2 - 4x + 20$$

SAME RESULT, DIFFERENT FORM.

PARABOLA

150

EQUATION LOCUS P
EQUIDISTANT FROM Q (1,3)
AND X-AXIS



$$y^2 = (3-y)^2 + (x-1)^2$$

$$y^2 = 9 - 6y + y^2 + x^2 - 2x + 1$$

$$6y = x^2 - 2x + 10 \text{ (PARABOLA)}$$

$x^2 = 8y$ DIRECTRIX,
FOCUS,
EQUATION TANGENT AT (-4,2)

VERTEX (0,0)
 $a=2$, FOCUS (0,2)
DIRECTRIX $y=-2$

$$y = \frac{x^2}{8}, y' = m \text{ TANGENT} = \frac{x}{4} = -1$$

$$2 = 4 + b, \text{EQ. } y = -x - 2$$

Y-INTERCEPT TANGENT
TO $x^2 = 8y$ AT (4,2)

TRANS FORM
 $y = x^2 - 4x + 3$ TO FIND
FOCAL LENGTH, V, F & DIRECTRIX

$$y = \frac{x^2}{8}, y' = \frac{x}{4} = 1 \text{ AT } (4,2)$$

$$2 = 4 + b$$

EQ. TANGENT $y = x - 2$

Y-INTERCEPT (0,-2)

COMPLETING SQUARES
 $(x-2)^2 = 1(y+1), a = \frac{1}{4}$
 $V(2,1)$, CHECK $x = \frac{-b}{2a}, \checkmark$
FOCUS $(2, \frac{5}{4})$ DIR. $y = -1 \frac{1}{4}$

VERTEX (-1,-1) PARABOLA
THROUGH (0,0)

VERTEX (-1,-1)
PARABOLA THROUGH (0,0)

$$(x+1)^2 = 4a(y+1)$$

SUB (0,0), $4a=1$

$$y = x^2 + 2x$$

COMPARE

$$y = ax^2 + bx + c$$

SUB (0,0), $c=0$

$$\frac{-b}{2a} = -1, a = \frac{1}{2}b$$

SUB (-1,-1), $b=2$

$$y = x^2 + 2x$$

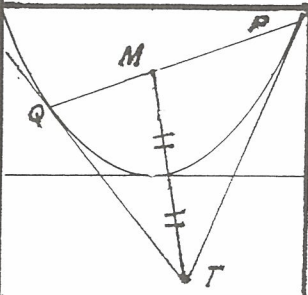
PARABOLA

151.

$P(2ap, ap^2), Q(2aq, aq^2)$ ON $x^2 = 4ay$
STANDARD PARABOLA

TANGENTS INTERSECT AT T . FIND MPQ . SHOW CURVE BISECTS TM

PARAMETRIC FORM: x & y ARE LINKED VIA
A 3RD. VARIABLE (PARAMETER)



$$y = \frac{x^2}{4a} \longrightarrow y' = \frac{x}{2a}$$

$$y' = m_{\text{TANGENT}} = p \text{ AT } P$$

EQUATION TANGENT AT P

$$ap^2 = 2ap^2 + b$$

$$y_p = px - ap^2 \quad (1)$$

$$y' = m_{\text{TANGENT}} = q \text{ AT } Q$$

EQUATION TANGENT AT Q

SIMILARLY

$$y_q = qx - aq^2 \quad (2)$$

AT T , THEY ARE EQUAL:

$$x(p - q) = a(p^2 - q^2)$$

$$x = a(p + q), \text{ SUB (1), } y = apq \therefore T \{ a(p + q), apq \}$$

$$MPQ = \left\{ a(p + q), \frac{a}{2}(p^2 + q^2) \right\} \therefore \underline{MMT} = \left\{ a(p + q), \frac{a}{4}(p + q)^2 \right\}$$

AVERAGE OF M & T (MIDPOINT FORMULA)

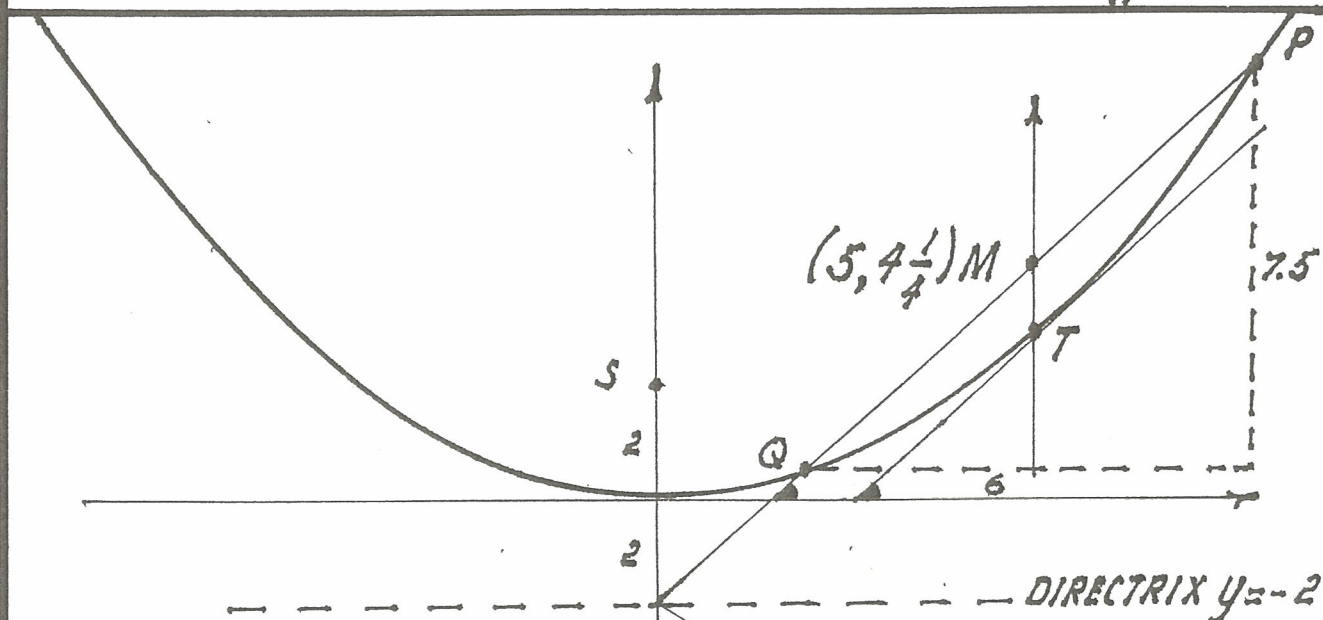
SUB $x^2 = 4ay$, TRUE! q.e.d.

PARABOLA

152.

$P(8,8), Q(2, \frac{1}{2})$ ON $x^2 = 8y$. EQUATION PQ

SHOW MPQ LIES ON $x^2 = 4(y+2)$. PQ & AXIS MEET ON THE DIRECTRIX.
LINE THROUGH M $\parallel$ AXIS MEETS PARABOLA AT T. SHOW TANGENT AT T $\parallel$ PQ.



$M(5, 4\frac{1}{4})$. SUB $x^2 = 4(y+2)$, TRUE

$$m_{PQ} = \frac{7.5}{6} = \frac{5}{4}. \text{ USE P: } 8 = 10 + b$$

EQUATION PQ. $y = \frac{5}{4}x - 2$ TRUE

EQUATION LINE THROUGH M $\parallel$ AXIS: $x = 5$

SUB $x^2 = 8y$. $T(5, \frac{25}{8})$

$$\downarrow y = \frac{x^2}{8}, y' = \frac{x}{4} = \frac{5}{4} \text{ AT } T$$

SAME GRADIENT AS PQ $\therefore$ TANGENT AT T $\parallel$ PQ

PROBABILITY

153.

EQUALLY LIKELY OUTCOMES

PACK (DECK) OF ⁵²CARDS
4 SUITS

13 CLUBS	BLACK	13 SPADES	
13 DIAMONDS		13 HEARTS	RED

TOSSING A (FAIR) COIN
CHANCE OF TOSsing
A HEAD is $\frac{1}{2}$. A TAIL is $\frac{1}{2}$

ACE

$\frac{4}{52}$ $\frac{1}{13}$

ROLLING AN UNBIASED DIE
CHANCE OF THROWING
A 4 is $\frac{1}{6}$

RED OR BLACK
ACE

$\frac{2}{52}$ $\frac{1}{26}$

DRAWING A MARBLE
FROM A BAG WITH 3 RED AND
2 BLACK ONES $\frac{3}{5}$
 $\frac{2}{5}$

ACE OF CLUBS

$\frac{1}{52}$

CLUB

$\frac{13}{52}$ $\frac{1}{4}$

RANGE

100
BOXES

PROBABILITY
OF SCORES

1 CERTAIN
LIKELY
 $\frac{1}{2}$ EVEN CHANCE
UNLIKELY
● IMPOSSIBLE

Number of matches per box	Frequency	Relative frequency
---------------------------	-----------	--------------------

< 50

47 7 0.07

48 12 0.12

49 20 0.20

50 35 0.35

51 19 0.19

52 6 0.06

53 1 0.01

CHANCE .39
OR $\frac{39}{100}$

CHANCE
OF SELECTING
A BOX
WITH 50

PROBABILITY ONE

DRAWING A RED MARBLE
FROM A BAG WITH RED ONES

PROBABILITY ZERO

DRAWING A BLACK ONE

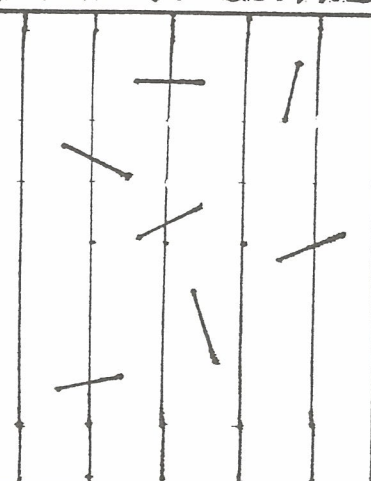
.35 OR $\frac{7}{20}$

PROBABILITY

154.

COMPTE DE BUFFON (1760)
EXPERIMENT WITH x cm
STICK AND x LINES x cm
APART

DROP THE STICK
200 TIMES ONTO
THE LINED PAPER
FROM A SUITABLE HEIGHT



CHECK
WHETHER
THE (CROSS LINE)
PROBABILITY
IS $\frac{2}{x}$
IT MIGHT
PROVE THAT
THE EARTH IS ROUND

TWO DICE
ROLLED SIMULTANEOUSLY.

PRODUCT RULE

36 COMBINATIONS

CHANCE OF THROWING
A TOTAL OF 5

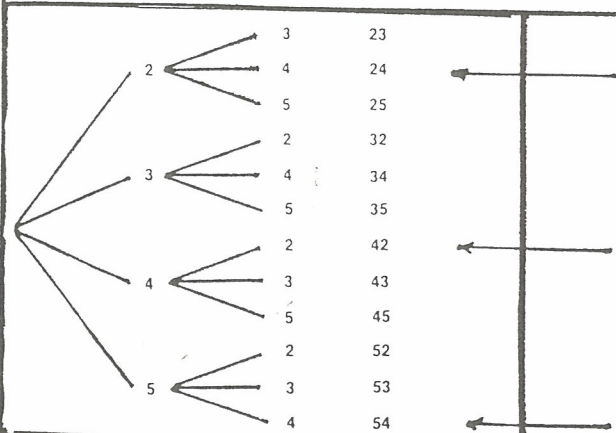
(1, 4)
(4, 1)
(2, 3)
(3, 2)

$$\frac{4}{36} = \frac{1}{9}$$

TREE DIAGRAM

2, 3, 4, 5 WRITTEN ON CARDS

DRAW 2 CARDS TO GIVE
A 2-DIGIT NUMBER

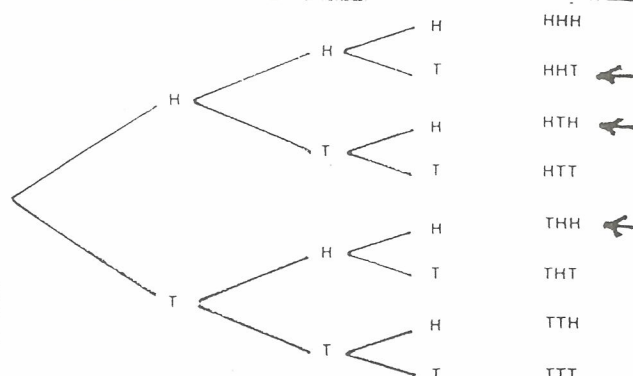


4x3 POSSIBLE COMBINATIONS

NUMBERS DIVISIBLE BY 6 $\frac{1}{4}$

A COIN TOSSED
3 TIMES (OR 3 COINS ONCE)

2 HEADS 1 TAIL
IN ANY ORDER



$2^3 = 8$ COMBINATIONS

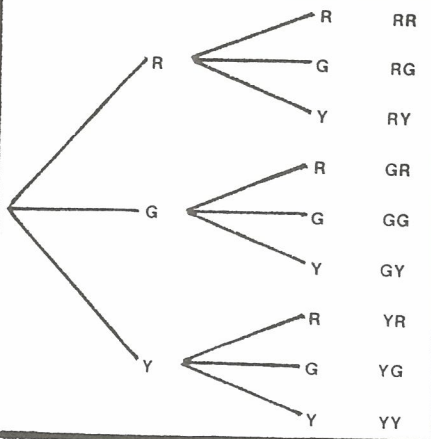
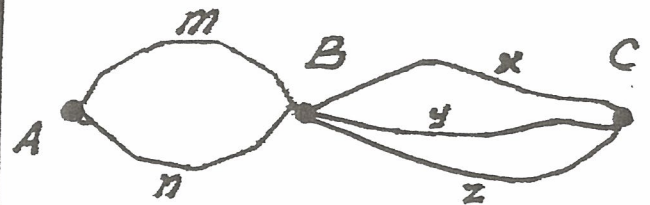
PROBABILITY
 $\frac{3}{8}$

PROBABILITY

155.

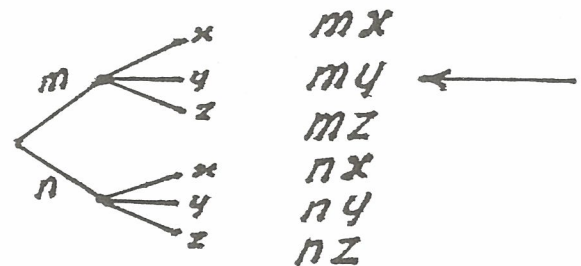
POTS: RED, GREEN & YELLOW
ORDER 2 WITHOUT SPECIFICATION

CHANCE GETTING ONE COLOUR



←
 $3 \times 3 = 9$
 COMBINATIONS

THERE ARE 6 WAYS
 TO GO FROM A TO C
 VIA B



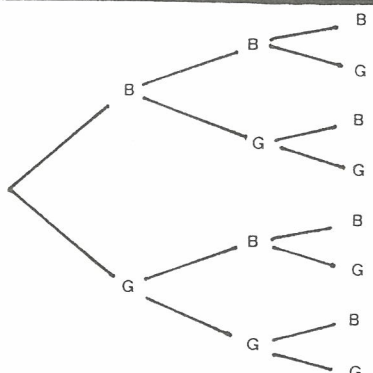
PROBABILITY $\frac{1}{3}$

PROBABILITY OF CHOOSING
 $my = \frac{1}{6}$

FAMILY OF 3 CHILDREN

3 GIRLS	2 BOYS 1 GIRL	ELDEST GIRL	YOUNGEST BOY
------------	------------------	----------------	-----------------

BIASED DIE



$2^3 = 8$
 COMBINATIONS

THERE ARE 7 POSSIBILITIES
 IF THROWING A 6
 IS TWICE AS LIKELY
 AS ANY OTHER NUMBER

$\frac{1}{8}$

$\frac{3}{8}$

$\frac{1}{2}$

$\frac{1}{2}$

CHANCE 3

CHANCE 6

$\frac{2}{7}$

$\frac{1}{7}$

PROBABILITY

156.

CRAPS

ROLLING 2 DICE

7 OR 11 WIN

2, 3 OR 12 LOSE

ANY OTHER TOTAL: PLAYERS POINT
KEEP ROLLING: REPEAT BEFORE 7 WIN

PROBABILITY TREE

3 WHITE 4 BROWN
NOT REPLACED

TOTAL WAYS 36 OUTCOMES

2

1

FIRST THROW

3

2

2, 3 OR 12

$\frac{1}{9}$

4

3

5 BEFORE 7

$\frac{1}{9}$ $\frac{1}{6}$

5

4

WIN OR LOSE

$\frac{4}{36}$ $\frac{9}{36}$

6

5

END ↑ ADD

$\frac{1}{3}$

7

6

NOT END

$\frac{2}{3}$

8

5

NO 7, 11, 2, 3 OR 12 6 OR

9

4

BEST TOTAL

8

10

3

10: ANY THROW

$\frac{1}{12}$

11

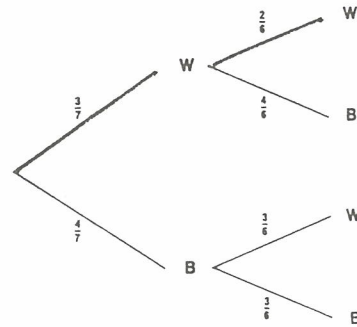
2

7 & 11 MOST DESIRABLE

12

1

THEN 6 8 5 9 4 10



CONDITION

WHITE

FIRST

TIMES

PLUS

ROUTINE

$$WW \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

7X6 = 42 POSSIBLE OUTCOMES

3X2 = 6 WAYS

$$\frac{1}{7} + \frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \text{ONE} \checkmark$$

COIN AND DIE

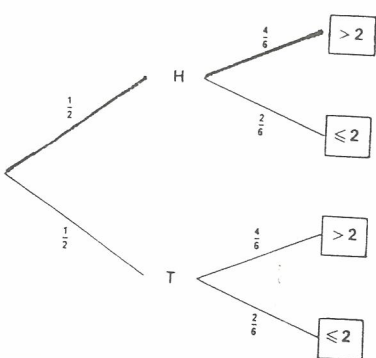
PROBABILITY HEAD AND
A NUMBER 7 2

9 CARDS

2 WITH NO. 5

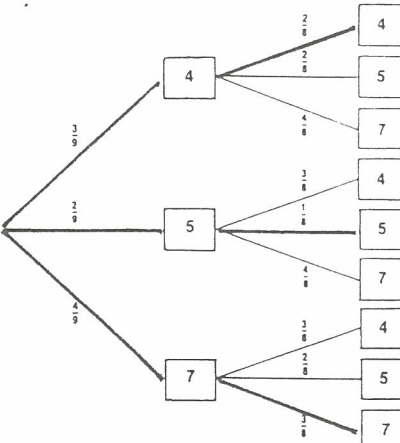
3 WITH NO. 4

4 WITH NO. 7



PRODUCT

RULE



2

DRAWS
TO FORM
2-DIGIT
NUMBERS

NO

REPLACING

$$\frac{1}{2} \times \frac{4}{6} = \frac{1}{3}$$

CHANCE 44

$\frac{1}{12}$

CHANCE
DOUBLE $\frac{5}{18}$

$$\frac{6}{72} + \frac{2}{72} + \frac{12}{72}$$

PROBABILITY

157

COMPLEMENTARY EVENTS

FAMILY OF 5 CHILDREN
AT LEAST 1 BOY

NO BOYS $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
 $= \frac{1}{32}$

AT LEAST 1 COMPLEMENT $31/32$

3 THROWS OF A DIE
AT LEAST ONE 6

NO SIX
 $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

AT LEAST ONE
 IS THE
 COMPLEMENT
 OF NONE $\frac{91}{216}$

NON MUTUALLY EXCLUSIVE EVENTS

OVERLAPPING EVENTS

NUMBERS 1-10
ON CARDS

INTERSECTION

DRAWING LESS THAN 5
OR DIVISIBLE BY 2

$$\frac{4}{10} + \frac{5}{10} - \frac{2}{10} = \frac{7}{10}$$

MUTUALLY EXCLUSIVE EVENTS

DISJOINT SETS (UNION)

PACK OF 52 ^{PLAYING} CARDS

NUMBERS 1-10
ON CARDS

RED OR 10
NOT MUTUALLY EXCLUSIVE

CHANCE OF DRAWING

LESS THAN 5 OR
DIVISIBLE BY 7

THERE ARE 2 RED TENS.

THEY MUST NOT BE
COUNTED TWICE.

$$\frac{4}{10} + \frac{1}{10} = \frac{1}{2}$$

$$\frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{7}{13}$$

PROBABILITY

158.

5 BLACK 4 WHITE	WITH REPLACEMENT	5 BLACK 4 WHITE	WITHOUT REPLACEMENT
	BB $\frac{25}{81}$ BW $\frac{20}{81}$ WB $\frac{20}{81}$ WW $\frac{16}{81}$ TOTAL ONE ✓		BB $\frac{20}{72}$ BW $\frac{20}{72}$ WB $\frac{20}{72}$ WW $\frac{12}{72}$ TOTAL ONE ✓
BOTH BLACK	$\frac{25}{81}$	BOTH BLACK	$\frac{5}{18}$
SAME COLOUR	$\frac{41}{81}$	SAME COLOUR	$\frac{4}{9}$
DIFFERENT	$\frac{40}{81}$	DIFFERENT	$\frac{5}{9}$

FRENCH

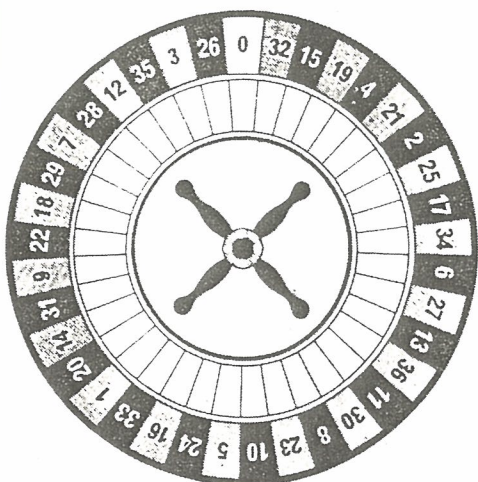
ROULETTE

ONE
ZERO

SPIN THE WHEEL
THROW MARBLE IN OPPOSITE DIRECTION

FAITES VOS JEUX
RIEN NE VA PLUS

PLAY
STOP



TRANS

CARRE
GROUP
OF 4

EN PLEIN

COLONNE

DOUZAINNE

COLONNES

Passer high 19 36	1	2	3	Manque low 1 18
	4	5	6	
	7	8	9	
Pair all the even numbers	10	11	12	Impair all the odd numbers
	13	14	15	
	16	17	18	
	19	20	21	
Non (black ♦)	22	23	24	Rouge (red) ◇
	25	26	27	
	28	29	30	
	31	32	33	
	34	35	36	

SIX AINE
6

VERSALE

CHEVAL
2

SINGLE

COLUMN

DOZEN

FRENCH

37 SLOTS

AMERICAN

38 SLOTS

PROBABILITY

159.

BETTING

ODDS 5 to 1
AGAINST

MEANS PROBABILITY OF
WINNING IS $\frac{1}{6}$

\$100 BET

WIN 5×100 \$500
+ STAKE MONEY 100

COLLECT \$600

BOOK MAKER

ODDS 7 to 2
AGAINST

MEANS CHANCE OF
WINNING IS $\frac{2}{9}$

\$30 BET

WIN $7\frac{1}{2} \times 30$ \$105
+ STAKE MONEY 30

COLLECT \$135

PROBABILITY OF

WINNING 2 IN 7

MEANS 2 WIN 5 LOSE
ODDS 5 TO 2

\$10 BET

WIN $5/2 \times 10$ \$25
+ STAKE MONEY 10

COLLECT \$35

INVEST \$x WIN

ODDS 4 to 1
AGAINST

MEANS PROBABILITY OF
WINNING IS 1 IN 5

WIN $4x = 100$
STAKE $x = 25$

PROBABILITY

160.

Evens **ODDS 1-1** **PROB. $\frac{1}{2}$**

ODDS ON 3 to 1

POKER MACHINE
3 WHEELS
20 SYMBOLS EACH

MEANS PROBABILITY OF
WINNING IS $\frac{3}{4}$

MEANS

ODDS AGAINST 1 to 3

BET \$30

WIN $\frac{1}{3} \times 30$ \$10

+ STAKE MONEY 30

COLLECT \$40

SYMBOL	1	2	3
APPLE	5	4	1
PEAR	5	4	2
LEMON	3	3	3
ACE	3	3	4
JACK	2	3	5
KING	2	3	5

PRODUCT RULE

3 APPLES 20 WAYS

L. L. ACE 36 WAYS

K. K. ANYTHING. $2 \times 3 \times 2$ 20 WAYS

3 JACKS OR 3 APPLES $30 + 20$ WAYS

PEAR, APPLE, JACK $\frac{1}{4} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{80}$
PROBABILITY

OLD NUMBERPLATES
NEW NUMBERPLATES

PERMUTATIONS
ORDER IS IMPORTANT

OLD 3 LETTERS
3 DIGITS

$26 \times 26 \times 26 \times 10 \times 10 \times 10$
POSSIBILITIES

TRIFECTA IN HORSE RACING
1st. 2nd. 3rd. IN ORDER

NEW 4 LETTERS
2 DIGITS

$26 \times 26 \times 26 \times 26 \times 10 \times 10$
COMBINATIONS

10 HORSES

10 CAN COME FIRST

9 CAN COME SECOND

8 CAN COME THIRD

120 PERMUTATIONS

THEORETICAL CHANCE OF
WINNING $\frac{1}{120}$ WITH EQUAL ABILITY

PROBABILITY

161.

50 LOTTERY TICKETS

PERMUTATIONS ON THE CALCULATOR

ANY OF THE
50 CAN COME FIRST

TRIFECTA
8 HORSE RACE $8P_3 = 336$
8X7X6 WAYS

ANY OF THE OTHER
49 CAN COME SECOND

TRIFECTA
12 HORSE RACE $12P_3 = 1320$
12X11X10
PROBABILITY
OF WINNING $1/1320$

2450 WAYS TO WIN
THE DOUBLE

PROBABILITY $1/2450$

5 PEOPLE CAN
LINE UP IN $5P_5 = 120$ WAYS
5X4X3X2X1

CALLED
FACTORIALS $5!$ QUICKER
120

COMBINATIONS ORDER IS NOT IMPORTANT

COMBINATIONS LOTTO SOCCER POOLS

7 TENNIS PLAYERS
TO PLAY DOUBLES $7C_2 = 35$ SHORT

LONGWAY:
CANCELLING THE
DOUBLES

$$\frac{7P_4}{4!}$$

$$40C_6 =$$

3,838,380
WAYS

8 GAMES
OUT OF 55
 $55C_8 = 1217566350$
WAYS

FIRST
2 HORSES OUT OF 10

$$10C_2 = 45$$

QUINELLAS

CHANCE
TO WIN
APPR.

CROSS OFF
11: $11C_8 = 165$
SELECTIONS

4 CARDS FROM 52
PROBABILITY 4 ACES
(CHECK) ✓

$$52C_4 = 270725$$

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}$$

1 IN 4,000,000
FOR 4 TICKETS
REAL PROB $1/\text{MILLION}$

PROBABILITY
MAX. 24 POINTS
APPR. 1 IN 7 MILLION

PROBABILITY

162.

INTEGER n

$$1 \leq n \leq 144$$

PICKED AT RANDOM. PROB. n^2

10 PEOPLE: 5 FRENCH

3 GERMAN, 2 ENGLISH

2 SELECTED AT RANDOM

THERE ARE
12 SQUARES IN THE
INTERVAL

PROBABILITY $\frac{1}{12}$

2 FRENCH

$$\frac{1}{2} \times \frac{4}{9} = \frac{2}{9}$$

1 F & 1 G

$$I \text{ FG} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$II \text{ G.F} = \frac{3}{10} \times \frac{5}{9} = \frac{1}{6}$$

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

NEITHER F.G.

$$\frac{1}{5} \times \frac{1}{9} = \frac{1}{45}$$

CHOOSE ANY LETTER
ON A PRINTED PAGE
COMMENT
AT RANDOM. PROB. $e = \frac{1}{26}$

EGGS 90% FRESH
PICK 3

MEANS

LETTERS IN THE ALPHABET
ARE
NOT USED
WITH EQUAL FREQUENCY

PROB. e FAR GREATER
THAN $\frac{1}{26}$

PROB. ALL FRESH

PROB. FRESH $\frac{9}{10}$

PROB. STALE $\frac{1}{10}$

IGNORE

NOT BEING
REPLACED

$$.9^3 = 72.9\%$$

INDEPENDENT TRIALS
NO SPECIFIED NUMBER

AT LEAST
PROB. 1 STALE

AT LEAST ONE IS THE
COMPLEMENT OF NONE

$$100 - 72.9 = 27.1\%$$

PROBABILITY

163.

FOUR DIFFERENT PAIRS
OF SOCKS
 SELECT 2 AT RANDOM
socks
 PROB. MATCHING PAIR

COMMENT: 2 COINS TOSSED.

H.H & T.T & HT. PROB. $\frac{1}{3}$

3 COINS TOSSED. PROB. 2H & 1T

1ST. CAN BE ANY
 2ND $\frac{1}{7}$ = REQD. PROB

4 POSSIBILITIES
 HH. HT. TH. TT.

12 TEAMS PROB. WIN
 COMPETITION $\frac{1}{12}$. COMMENT

$$\text{PROB. HH} = \frac{1}{4}$$

NOT EQUALLY LIKELY
 EVENTS ALL PLAYERS
 ARE DIFFERENT

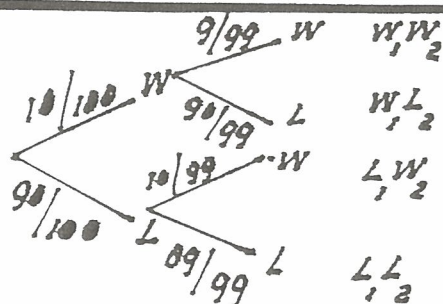
HHH. HHT. HTH. HTT. T HH.
 THT. TTH. TTT (TREE DIAGRAM)

$$\text{PROB. 2H & 1T} = \frac{3}{8}$$

100 RAFFLE TICKETS
 1ST. & 2ND PRIZE
 10 TICKETS
 BOUGHT

CLASS A: 11 F & 9 M | SELECT
 CLASS B: 8 F & 12 M | A & B

PROB. 2M, 2F, 1M & 1F



$$\frac{9}{20} \times \frac{12}{20} = \frac{27}{100}$$

$$\frac{11}{20} \times \frac{8}{20} = \frac{11}{50} = \frac{22}{100}$$

$$W_1 W_2 = \frac{1}{10} \times \frac{9}{99} = \frac{1}{110}$$

$$W_1 = \frac{1}{10}$$

$$L_1 L_2 = \frac{9}{10} \times \frac{89}{99} = \frac{89}{110}$$

3 POSSIBILITIES: 2M, 2F, 1M & 1F
 $\therefore 1M & 1F = 1 - \frac{49}{100} = \frac{51}{100}$
 COMPLEMENT

$$\text{AT LEAST ONE} = 1 - \frac{89}{110} = \frac{21}{110}$$

PROBABILITY

164

4 BLACK, 3 WHITE		SIX CARDS A-F EITHER E OR A 3RD	COMMENT ROAD TOLL
2 W	W & B	NO REPLACEMENT	1 IN 4000
$\frac{3}{7} \times \frac{2}{6}$	$\frac{3}{7} \times \frac{4}{6} +$ $\frac{4}{7} \times \frac{3}{6} = \frac{4}{7}$	EQUALLY LIKELY CHANCE FOR ALL CARDS TO BE THIRD $\therefore \text{PROB} = 2 \times \frac{1}{6} = \frac{1}{3}$	NOT A PROBABILITY FOR EVERYONE. NOT EVERYONE IS ON THE ROAD AT ANY ONE TIME.
FLOWER SEEDS. PROB. PINK FLOWER $\frac{1}{4}$ LEAST n TO GET AT LEAST 1, .99		BULBS. 5 GOOD, 3 BAD. PICK 2. 1ST B, B & B, G & G, B & G	
PROB. NO PINK $\frac{3}{4}$		FIRST BAD	$\frac{3}{8}$
AT LEAST ONE IS THE COM- PLEMENT OF NONE.		BOTH BAD	$\frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$
$1 - \left(\frac{3}{4}\right)^n > .99$	$\left(\frac{3}{4}\right)^n < .01$	BOTH GOOD	$\frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$
$n(\log_3 3 - \log_3 4) < -2$		BAD GOOD	$\left(\frac{3}{8} \times \frac{5}{7}\right) +$
$n > 16.007$	LEAST $n=17$	GOOD BAD	$\left(\frac{5}{8} \times \frac{3}{7}\right) = \frac{15}{28}$
2 DICE: BOTH 6, 1 & 6, AT LEAST ONE WITH 1, TOTAL 6		CHESS W(START) WIN .5 B WIN = .3 . 2 GAMES W, B WIN = 1, DRAW = $\frac{1}{2}$ POINT	
DOUBLE 6 1 IN 36		DRAW $1 - (.5 + .3) = .2$	
$(1 \& 6) \& (6 \& 1) 2 \times \frac{1}{36} = \frac{1}{18}$		2 POINTS IS W WIN, B WIN .5 x .3 = .15	
11 12 13 14 15 16	$P \frac{11}{36}$ AT LEAST ONE /	1.5, 5.1, 2.4, 4.2 3.3 $P \frac{5}{36}$	$< \frac{1}{2} = \text{COMPL. OF AT LEAST } \frac{1}{2}$ $= WN + WD + DW =$ $1 - (.15 + .10 + .06) = .69$

QUADRATIC EQUATIONS

165.

$$3x^2 - 12 = 0$$

$$2x^2 - 6x = 0$$

THINK
FACTORISE BINOMIAL

START: $\cdot ()$

$$3(x^2 - 4) = 0$$

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

IF YOU DON'T START THE FIRST STEP,
YOU CAN'T CLIMB TO THE TOP.

FACTORISE

START $\cdot ()$

$$2x(x - 3) = 0$$

$$x = 0$$

$$x = 3$$

ANSWERS ARE CALLED
ROOTS, ZEROS, X-INTERCEPTS
OF THE PARABOLA
 $y = 2x^2 - 6x$

$$(x - 2)(x + 5) = 0$$

TYPE 1

$$x^2 - x - 12 = 0$$

ROUTINE

$$x = 2$$

$$x = -5$$

BECAUSE THESE VALUES
SATISFY THE EQUATION

TO FACTORISE, FIND
2 NUMBERS, PRODUCT - 12
SUM - 1

STEP 3:

$$(x - 4)(x + 3) = 0$$

$$x = 4$$

$$x = -3$$

START
THE
ROUTINE:
MEANS
 $(x)(x)$

THEN:

$$(x -) (x +)$$

MEANS $\uparrow \quad \uparrow$

NEGATIVES 'WIN BY 1'
KEEP MOVING.

QUADRATIC EQUATIONS

166.

TYPE 2

PRIME

$$2x^2 - 5x - 12 = 0$$

$$\begin{array}{cc} (2x & +3) \\ & \times \\ (x & -4) \end{array}$$

$$(2x+3)(x-4)=0$$

$$x = -1\frac{1}{2}$$

$$x = 4$$

TYPE 3

COMPOSITE

$$6x^2 + 11x - 4 = 0$$

$$\begin{array}{cc} (3x & -1) \\ & \times \\ (2x & +4) \end{array}$$

$$(3x-1)(2x+4)=0$$

$$x = \frac{1}{3}$$

$$x = -2$$

ALL PURPOSE

RECIPE

THE FORMULA

ANSWERS

IN SURD FORM

$$2x^2 - 3x - 7 = 0$$

GIVES THE ROOTS DIRECTLY

GENERAL EQUATION

$$ax^2 + bx + c = 0$$

INDICATES

FORMULA OR

COMPLETING THE SQUARE

$$a=2$$

$$b=-3$$

$$c=-7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

b FOR begin (pointing to -b)
all clear (pointing to the rest of the formula)
2x2=4 (pointing to the 4 in 4ac)

VISUALISE

$$x = \frac{3 \pm \sqrt{65}}{4}$$

QUADRATIC EQUATIONS

167

$$3x^2 - 10x + 8 = 0$$

$$x^2 - 2x - 8 = 0$$

FORMULA

NO NEED TO FACTORISE

**COMPLETE
THE SQUARE**

$$a=3 \quad b=-10 \quad c=8$$

$$x = \frac{10 \pm \sqrt{4}}{6}$$

$$x = 2$$

$$x = 1\frac{1}{3}$$

HALVE & SQUARE, THEN ADJUST

$$x^2 - 2x + 1 = 9$$

$$(x-1)^2 = 9 \therefore x-1 = \pm 3$$

$$x = 4$$

$$x = -2$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

CHECK $x=-1, x=3$

SOLVE GRAPHICALLY

REWRITE AS

$$x^2 = 2x + 3$$

PARABOLA = STRAIGHT LINE

MEANS AT THEIR

INTERSECTIONS

x - AND y -VALUES

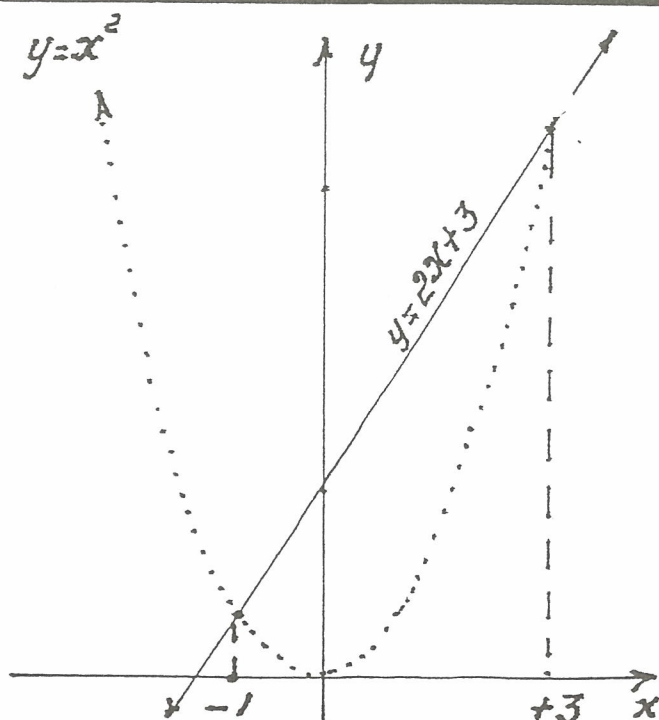
ARE EQUAL

DRAW

$$y = x^2$$

AND

$$y = 2x + 3$$



QUADRATIC EQUATIONS

168.

FROM ROOTS
TO EQUATION $x = -2$
 $x = 3$

SUM
 $\alpha + \beta$

PRODUCT
 $\alpha \beta$

THE ROOTS ARE ALSO
CALLED α AND β
alpha beta

$$(x - \alpha)(x - \beta) = 0$$

$$= x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$= x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

HENCE
 $(x - \alpha)(x - \beta) = 0$
 $\alpha = -2$ $\beta = 3$

QUADRATIC IDENTITIES
(REVISED GENERAL FORMULA)

$$(x + 2)(x - 3) = 0 \quad x^2 - x - 6 = 0$$

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

$$2x^2 - 5x + 1 = 0$$

ROOTS $3 \pm \sqrt{5}$

$$9^x - 28 \cdot 3^x + 27 = 0$$

$$\alpha + \beta = 2\frac{1}{2}, \alpha\beta = \frac{1}{2}$$

SUM 6
PRODUCT 4

$$3^{2x} - 28 \cdot 3^x + 27 = 0$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = 5$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad (3^x - 27)(3^x - 1) = 0$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 5\frac{1}{4}$$

$$x^2 - 6x + 4 = 0$$

$$x = 3, x = 0$$

$$2x^2 - (m + 2)x + m = 0 \quad \text{ROOTS } \alpha, 2\alpha$$

SUM 3α

$$m + 2 = 6\alpha$$

$$2\alpha^2 - 3\alpha + 1 = 0$$

PRODUCT $2\alpha^2$

$$m = 4\alpha^2$$

$$(2\alpha - 1)(\alpha - 1) = 0$$

$$\alpha = 1$$

$$m = 4$$

$$\alpha = \frac{1}{2}$$

$$m = 1$$

QUADRATIC EQUATIONS

169

$$2x^2 - 5x + 1 = 0$$

FIND $(\alpha-3)(\beta-3)$

$$x^2 - (2+k)x + 3k = 0$$

$$\alpha + \beta = 2\frac{1}{2}$$

$$\alpha\beta = \frac{1}{2}$$

$$\alpha + \beta = 2+k$$

$$\alpha\beta = 3k$$

$$(\alpha-3)(\beta-3)$$

$$\alpha\beta - 3(\alpha + \beta) + 9$$

IF $\alpha + \beta = 5, k = 3$

IF $\alpha\beta = 12, k = 4$

IF $\alpha\beta = 4(\alpha + \beta)$

SUBSTITUTE

$$\frac{1}{2} - 7\frac{1}{2} + 9 = 2$$

$$3k = 4(2+k)$$

$$k = -8$$

GENERAL
QUADRATIC
EQUATION

$$ax^2 + bx + c = 0$$

FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

THE
DISCRIMINANT

DELTA

GREEK  CAPITAL
D

$$b^2 - 4ac$$

IT DISCRIMINATES: SHOWS WHETHER THERE ARE 0, 1 OR 2 ROOTS

$$\Delta > 0$$

$$\Delta = 0$$

$$\Delta < 0$$

2 ROOTS

1 ROOT

NO ROOTS

$$3x^2 - 10x + 8 = 0$$

$$x^2 - 6x + 9 = 0$$

$$-2x^2 + x - 5 = 0$$

$$\Delta = 4$$

$$\Delta = 0$$

$$\Delta = -39$$

QUADRATIC EQUATIONS

170.

EQUAL ROOTS

$$3x^2 - 8x + K = 0$$

$$64 - 12K = 0$$

$$K = 5\frac{1}{3}$$

NO REAL ROOTS

$$ax^2 + 3x - 4 = 0$$

$$9 + 16a < 0$$

$$a < -\frac{9}{16}$$

EQUAL ROOTS MEANS ONE ROOT

$$\Delta = 0$$

$$x = \frac{8}{6} = 1\frac{1}{3}$$

NEGATIVE DEFINITE

$$2x^2 - 5x - K = 0$$

ROOTS
EQUAL

$$\Delta = 0$$

$$25 + 8K = 0$$

$$K = -3\frac{1}{8}$$

REAL

$$\Delta \geq 0$$

$$25 + 8K \geq 0$$

$$K \geq -3\frac{1}{8}$$

NOT REAL

$$\Delta < 0$$

$$25 + 8K < 0$$

$$K < -3\frac{1}{8}$$

DIFFERENT

$$\Delta > 0$$

$$25 + 8K > 0$$

$$K > -3\frac{1}{8}$$

QUADRATIC EQUATIONS

171

ANALYSE THE DISCRIMINANT

$$4x^2 + 3Kx + K^2 = 0$$

$$9K^2 - 16K^2 = -7K^2$$

SQUARES ARE ≥ 0

$K > 0$	Δ NEGATIVE	NO REAL ROOTS
$K < 0$	Δ NEGATIVE	NO REAL ROOTS
$K = 0$	Δ NOUGHT	$x = 0$ ONE ROOT

ANALYSE

$$x + \frac{1}{x} = K \quad (\text{REAL ROOTS})$$

$$x^2 - Kx + 1 = 0 \quad \Delta = K^2 - 4$$

ROUTINE:
DISJOINT
GRAPH.
TEST
TO CONFIRM
 $K = 0$,
FALSE

THE EQUATION HAS REAL ROOTS

$$K^2 \geq 4$$

THE SET OF ALL K s SUCH THAT...

SOLUTION SET (TRUTH SET)
UNION

$$\{k: k \geq 2\} \cup \{k: k \leq -2\}$$

x REAL



QUADRATIC EQUATIONS

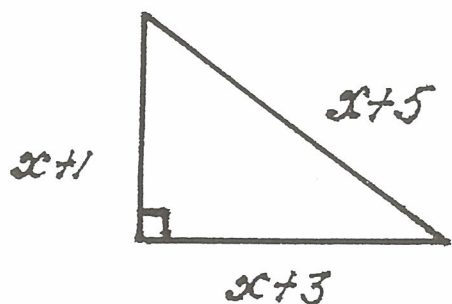
172.

RIGHT TRIANGLE

$(x+1) \quad (x+3) \quad (x+5)$

$$\alpha + \beta = 3$$

$$\alpha \beta = -2$$



α & β ARE THE ROOTS

OF THE QUADRATIC EQUATION

$$(x - \alpha)(x - \beta) = 0$$

$$\begin{aligned} (x+5)^2 &= (x+1)^2 + (x+3)^2 \\ x^2 + 10x + 25 &= x^2 + 2x + 1 + x^2 + 6x + 9 \\ x^2 - 2x - 15 &= 0 \end{aligned}$$

$(x-5)(x+3) = 0$	$x = 5$	$x = -3$ N.A.
------------------	---------	------------------

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

SUBSTITUTE

$$x^2 - 3x - 2 = 0$$

EQUATION
REDUCIBLE TO
QUADRATICS

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0$$

$$x = \pm 2$$

$$x = \pm 1$$

EQUATION
REDUCIBLE TO
QUADRATICS

$$(x-1)^4 + (x-1)^2 - 6 = 0$$

$$\text{LET } (x-1)^2 = a$$

$$a^2 + a - 6 = 0$$

$$(a+3)(a-2) = 0$$

$$a = -3$$

$$(x-1)^2 = -3$$

NO REAL ROOTS

$$a = 2$$

$$(x-1)^2 = 2$$

$$x = 1 \pm \sqrt{2}$$

$$x^2 + x - 6 < 0$$

$$-x^2 - 3x + 4 < 0$$

$$(x+3)(x-2) < 0$$

TEST $x=0$, TRUE

ROUTINE



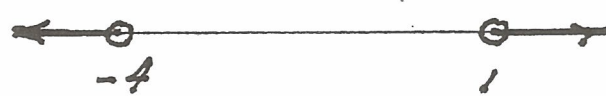
$$-3 < x < 2$$

$$x^2 + 3x - 4 > 0$$

$$(x+4)(x-1) > 0$$

TEST $x=0$, FALSE

ROUTINE



$$x < -4$$

$$x > 1$$

$$3x^2 + x - 2 \leq 0$$

$$-2x^2 + 3x + 14 \leq 0$$

$$(3x-2)(x+1) \leq 0$$

TEST $x=0$, TRUE



$$-1 \leq x \leq \frac{2}{3}$$

$$2x^2 - 3x - 14 \geq 0$$

$$(2x-7)(x+2) \geq 0$$

TEST $x=0$, FALSE



$$x \leq -2$$

$$x \geq 3\frac{1}{2}$$

QUADRATIC IDENTITIES

174.

$$3x^2 + 4x + 2 \equiv Ax(x-1) + B(x+1) + C$$

COEFFICIENTS OF LIKE POWERS
MUST BE EQUAL

TRUE FOR MORE THAN 2 VALUES
OF x

$$3x^2 + 4x + 2 \equiv Ax^2 - Ax + Bx + B + C$$

SUBSTITUTE
ANY CONVENIENT VALUE

$$3x^2 + 4x + 2 \equiv Ax^2 + x(B-A) + (B+C)$$

EQUATE COEFFICIENTS

$$A = 3$$

$$B - A = 4$$

$$B + C = 2$$

$$x = 1$$

$$x = 0$$

$$9 = 2B + C$$

$$2 = B + C$$

$$B = 7, C = -5$$

$$A = 3$$

$$B = 7$$

$$C = -5$$

$$x = -1$$

$$1 = 2A + C$$

$$A = 3$$

$$5x^2 + 2x - 3 \equiv A(x+1)^2 + B(x+1) + C$$

$$5x^2 + x - 2 \equiv a + b(x-1) + cx(x-1)$$

$$x = -1$$

$$C = 0$$

$$x = 0$$

$$\left. \begin{array}{l} -2 = a - b \\ 4 = a \end{array} \right\} b = 6$$

$$x = 0$$

$$-3 = A + B$$

$$x = 1$$

$$4 = a$$

$$x = -2$$

$$13 = A - B$$

$$x = 2$$

$$20 = a + b + 2c$$

$$A = 5, B = -8$$

$$C = 5$$

$$5x^2 + 2x - 3 \equiv 5(x+1)^2 - 8(x+1) + 0$$

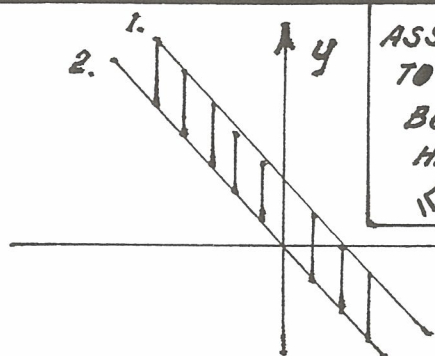
CHECK ✓

REGION

175.

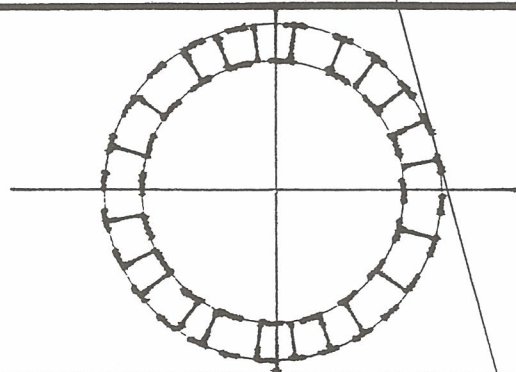
$$0 \leq x+y \leq 1$$

COMPOSITE INEQUALITY: TWO IN ONE



ASSUME EQUAL
TO ESTABLISH
BOUNDARIES.
HERE INCLUDED
< OR >

$$9 < x^2 + y^2 < 25$$



$$y \leq -x+1 \text{ (1) TEST (0,0) ✓}$$

$$y > -x \text{ (2) TEST (1,0) ✓}$$

 AREA IN BETWEEN. (1) & (2)
 INCLUDED

$$\text{CONCENTRIC CIRCLES (0,0)}$$

 RADII 3 & 5

$$\text{TEST (4,0), TRUE}$$

 AREA IN BETWEEN. BOUNDARIES
 NOT INCLUDED

$$\{(x,y): y \geq x\} \cap \{(x,y): 0 \leq y \leq 6\}$$

INTERSECTION

$$x+y \leq 3 \text{ (1)}$$

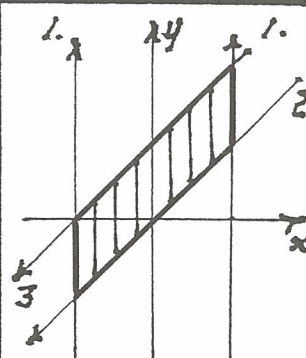
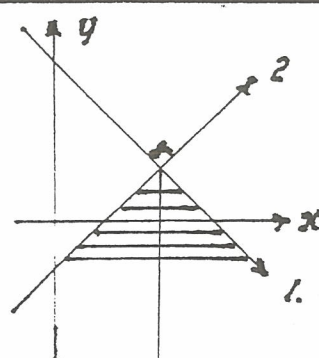
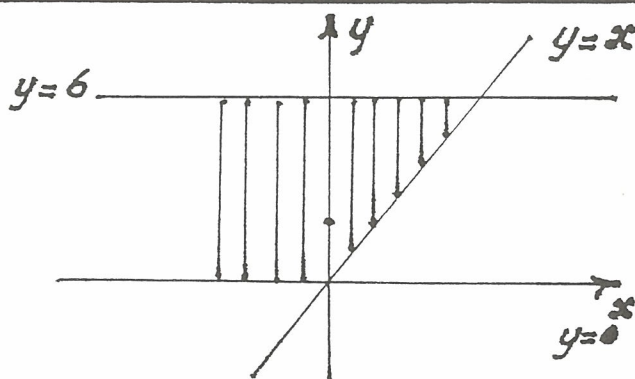
$$x-y \geq 1 \text{ (2)}$$

$$|x| \leq 1 \text{ (1)}$$

 $x \leq 1, x \geq -1$

$$y \geq x \text{ (2)}$$

 $(3) y \leq x+1$



$$y \geq x \text{ TEST (0,1) TRUE}$$

 AREA ABOVE $y = x$
 IN BETWEEN $y = 0$ & $y = 6$
 LINES INCLUDED

$$(1) \text{ TEST (0,0), TRUE}$$

$$(2) \text{ TEST (0,0), FALSE}$$

$$\text{SHADED AREA}$$

$$\text{LINES INCLUDED}$$

$$m_1 = -1 \text{ } \vee \text{ } m_2 = 1$$

$$\text{TEST (0, } \frac{1}{2} \text{)}$$

 TRUE

$$\text{SHADED AREA}$$

$$\text{LINES INCLUDED}$$

SCIENTIFIC (STANDARD) NOTATION

176

670 000 000

.00000067

THINK D.P
8 (PLACES BACK)
 6.7×10^8
↑ ONE COUNTING NUMBER
IN FRONT OF D.P.

THE EYES
↓
 6.7×10^{-7}
"COUNT" THE PLACES;
JUMPING OVER THEM IS SILLY!

2.34×10^6

2.34×10^{-6}

ONLY THE D.P. MOVES: DIGITS AND THEIR ORDER
REMAIN UNCHANGED

234 000

.0000234

$(3.2 \times 10^7) \times (7 \times 10^{-3})$

$(5.73 \times 10^6) \times (4.89 \times 10^1)$

WITHOUT CALCULATOR

$$22.4 \times 10^4 = 2.24 \times 10^5$$

↓ EXP. (CASIO)
EXPONENTIAL KEY

5.73 EXP.6 $\times$ 4.89 EXP.4 =
DISPLAY 2.80197" WRITE 2.8×10^1 "

$$\frac{2.1 \times 10^5}{7 \times 10^{-2}}$$

2.3×10^4

$$(8.4 \times 10^{13}) \div (7 \times 10^{-2}) = 1.2 \times 10^{15}$$

$$.3 \times 10^7$$

$$3 \times 10^6$$

WRITE
 2.3×10^4
INDICATES THAT THE CALC.
CAN DO UP TO
99

$$1.2 \times 10^{15}$$

$$x = 15$$

SEQUENCES AND SERIES

177

ARITHMETIC PROGRESSION A.P.

WITH COMMON DIFFERENCE

GEOMETRIC PROGRESSION G.P.

WITH COMMON RATIO

A.P				d	G.P			r
8	10	12		2	2	6	18	3
15	12	9		-3	16	12	9	$\frac{3}{4}$
4	$5\frac{1}{2}$	7		$1\frac{1}{2}$	3	-6	12	-2
8, 10, 12 10 IS THE ARITHMETIC MEAN. $(8+12) \div 2$				A.P. TEST	2, 6, 18 6 IS THE GEOMETRIC MEAN $\sqrt{2 \times 18}$			G.P. TEST
Term 1 a	T_2 $a+d$	T_3 $a+2d$	T_n $a+(n-1)d$		T_1 a	T_2 ar	T_3 ar^2	T_n $ar^{(n-1)}$
5	8	11	T_{20}		54	-18	6	T_7

$$a = 5$$

$$d = 3$$

$$T_{20} = 5 + 19 \times 3 = 62$$

$$a = 54$$

$$\frac{-18}{54} \rightarrow r = -\frac{1}{3}$$

$$T_7 = 54 \times \left(-\frac{1}{3}\right)^6 = \frac{2}{27}$$

CALCULATE
 3^6 FIRST

SEQUENCES AND SERIES

178.

A.P. $T_n = 3n + 5$

FIND FIRST FOUR

	$n \neq 0$	
T_1	$3 \times 1 + 5$	8
T_2	$3 \times 2 + 5$	11
T_3	$3 \times 3 + 5$	14
T_4	$3 \times 4 + 5$	17

$a = 8$ $d = 3$

G.P. $T_n = 2^n$

FIND FIRST FOUR

	$n \neq 0$
	2
	4
	8
	16

$a = 2$

$r = 2$

A.P. $T_5 = 31$, $T_{12} = 73$

G.P. $T_3 = 3$, $T_{10} = 6561$

SIMULTANEOUS EQUATIONS

$$\begin{aligned} T_{12} &= a + 11d = 73 \\ - T_5 &= a + 4d = 31 \end{aligned}$$

$$7d = 42$$

$$d = 6$$

$$a = 7$$

SIMULTANEOUS EQUATIONS

$$\begin{aligned} T_{10} &= ar^9 = 6561 \\ T_3 &= ar^2 = 3 \end{aligned}$$

$$r^7 = 2187$$

$$r = 2187^{\frac{1}{7}} = 3$$

$$a = \frac{1}{3}$$

SEQUENCES AND SERIES

179

A.P. 7, 12, 17, ...

WHICH TERM IS 372

G.P. 3, 6, 12, ...

WHICH TERM IS 384

$$a = 7$$

$$d = 5$$

$$a = 3$$

$$r = 2$$

EFFICIENCY:

SUBSTITUTE

FORMULAS

MENTALLY

MEMORY
TRAINING

$$T_n = 7 + (n-1)5 = 372$$

$$7 + 5n - 5 = 372$$

$$n = 74$$

$$T_n = 3 \times 2^{n-1} = 384$$

$$2^{n-1} = 128 = 2^7$$

$$n = 8$$

$$n-1 = \log 128 \div \log 2 = 7 \therefore n = 8$$

A.P. 3, 7, 11, ...

FIND THE FIRST TERM > 200

G.P. $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \dots$

SMALLEST TERM > 300

$$3 + (n-1)4 > 200$$

$$3 + 4n - 4 > 200$$

$$4n > 201$$

$$n > 50 \frac{1}{4}$$

$$T_{51} > 200$$

$$\frac{1}{2} \times 3^{n-1} > 300$$

$$3^{n-1} > 600$$

$$3^{n-1} > 3^5 \quad (\text{TRIAL})$$

$$3^6 = 729$$

$$n > 6 \therefore T_7 > 300$$

$$n-1 = \log 600 \div \log 3 \doteq 5.8$$

$$n > 6.8 \therefore T_7 > 300$$

SEQUENCES AND SERIES

180.

A.P. $\log 2, \log 4, \log 8$

FIND d & T_4, T_5

$$\log 4 = \log 2^2 = 2 \log 2$$

$$\log 8 = \log 2^3 = 3 \log 2$$

$$d = \log 2$$

$$T_4 = 4 \log 2 = \log 16$$

$$T_5 = 5 \log 2 = \log 32$$

G.P. \$100 @ 10% p.a.

COMPOUND. 4 YEARS

LONG WAY

$$1.1 \times 100 = \dots = \dots = \dots = \dots$$

AFTER 1 YEAR \$ 110.00

2 YEARS 121

3 YEARS 133.10

4 YEARS \$ 146.41

SHORTWAY $100 \times 1.1^4 = \$146.41$

SERIES $S_n = \text{SUM TERMS}$

$$\frac{1}{2}n(a + T_n) \quad \frac{1}{2}n\{2a + (n-1)d\}$$

T_n IS REPLACED BY 1 FOR last

$$S_n = a + (a+d) + \dots + (1-d) + 1$$

$$S_n = 1 + (1-d) + \dots + (a+d) + a$$

$$2S_n = (a+1) + (a+1) + \dots + (a+1) + (a+1)$$

$$2S_n = n(a+1) \therefore S_n = \frac{n}{2}(a+1)$$

$$\text{SUB. } 1 = a + (n-1)d$$

$$\therefore S_n = \frac{n}{2}\{2a + (n-1)d\}$$

SERIES $S_n = \text{SUM TERMS}$

$$r \neq 1 \quad \frac{a(r^n - 1)}{r - 1} \quad r < 1 \quad \frac{a(1 - r^n)}{1 - r}$$

$$S_n = a + ar + \dots + ar^{n-2} + ar^{n-1}$$

MULTIPLY BY r

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r < 1$$

SEQUENCES AND SERIES

181

ARITHMETIC SERIES

$$7+12+17+\dots$$

$$S_{20}$$

$$a=7$$

$$d=5$$

$$S_n = \frac{n}{2} \{2a + (n-1)d\}$$

$$10(14 + 19 \times 5)$$

$$S_{20} = 1090$$

GEOMETRIC SERIES

$$32+16+8+\dots$$

$$S_9$$

$$a=32$$

$$r = \frac{1}{2}$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad r < 1$$

$$S_9 = \frac{32(1-\frac{1}{2}^9)}{\frac{1}{2}}$$

$$= 64(1-\frac{1}{512}) = 63\frac{7}{8}$$

ONLY PUT 2^9 IN THE CALCULATOR!

ARITHMETIC SERIES

$$-15-8-1+6+\dots+188$$

FIND S_n

$$a=-15$$

$$d=7$$

$$l=188$$

$$-15 + (n-1)7 = 188$$

$$n-1 = 29$$

$$n = 30$$

$$S_{30} = \frac{n}{2}(a+l) = 15 \times 173 = 2595$$

GEOMETRIC SERIES

$$2+4+8+\dots+8192$$

FIND S_n

$$a=2$$

$$r=2$$

$$T_n=8192$$

$$2 \times 2^{n-1} = 8192$$

$$2^{n-1} = 4096 = 2^{12}$$

$$n = \ln 4096 \div \ln 2 + 1 = 13$$

$$n=13$$

$$S_{13} = \frac{a(r^{13}-1)}{r-1} = 16382$$

SEQUENCES AND SERIES

182.

ARITHMETIC SERIES

$$5 + 11 + 17 \dots$$

$$S_n = 208$$

$$a = 5$$

$$d = 6$$

$$S_n = 208$$

$$\frac{n}{2} \{10 + 6(n-1)\} = 208$$

$$n(5 + 3n - 3) = 208$$

$$3n^2 + 2n - 208 = 0$$

$$n = \frac{-2 \pm \sqrt{4 + 2496}}{6} = 8$$

GEOMETRIC SERIES

$$256 + 128 + 64 \dots$$

$$S_n = 511$$

$$a = 256$$

$$r = \frac{1}{2}$$

$$S_n = 511$$

$$\frac{256(1 - \frac{1}{2}^n)}{\frac{1}{2}} = 511$$

$$512(1 - \frac{1}{2}^n) = 511$$

$$1 - \frac{1}{2}^n = \frac{511}{512}$$

$$\frac{1}{2^n} = \frac{1}{512} = \frac{1}{2^9} \therefore n = 9$$

INSERT 3

ARITHMETIC MEANS
BETWEEN 2 AND 22

$$a = 2$$

$$T_5 = 22$$

$$2 + 4d = 22$$

$$d = 5$$

ARITHMETIC
PROGRESSION
(SEQUENCE)

$$2, 7, 12, 17, 22.$$

INSERT 3

GEOMETRIC MEANS
BETWEEN 3 AND 48

$$a = 3$$

$$T_5 = 48$$

$$3r^4 = 48$$

$$r = \pm 2$$

GEOMETRIC
PROGRESSIONS

$$3, 6, 12, 24, 48.$$

$$3, -6, 12, -24, 48$$

SEQUENCES AND SERIES

183.

ARITHMETIC MEAN
OF 2 POSITIVES x & y
is $\frac{x+y}{2}$

GEOMETRIC MEAN
 $\sqrt{xy}$

ARITHMETIC
MEAN



GEOMETRIC
MEAN

$$(\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy}$$

SQUARES ARE ALWAYS POSITIVE

$\therefore \frac{x+y}{2}$ IS GREATER THAN $\sqrt{xy}$

A.P. THE MEAN OF x & 3
EQUALS THE MEAN OF
 $3x$ & 8

G.P. 6 & 8 ARE THE
MEANS BETWEEN x & y

$$\frac{x+3}{2} = \frac{3x+8}{2}$$

$$2x = -5$$

$$x = -2\frac{1}{2}$$

$$x, 6, 8, y.$$

$$8x = 36$$

$$x = 4\frac{1}{2}$$

$$6y = 64$$

$$y = 10\frac{2}{3}$$

SEQUENCES AND SERIES

184.

LIMITING SUM OF A G.P.

WHEN r IS NUMERICALLY LESS THAN ONE

FOR $|r| < 1$,
WE USE

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim S = \frac{a}{1-r}$$

WHEN n IS APPROACHING ∞
THE VALUE OF r^n MAY BE
IGNORED. $\therefore \lim(1-r^n) = 1$

IF YOU HALVE A MARS BAR
AND KEEP ON HALVING THE
LEFT-OVER HALVES, YOU WILL
NEVER EVER HAVE TO BUY
ANOTHER MARS BAR.

GEOMETRIC SERIES

$$50 + 30 + 18 \dots$$

$$a = 50$$

$$r = \frac{3}{5}$$

$$\begin{aligned} \lim S &= \frac{50}{2/5} \\ &= 125 \end{aligned}$$

GEOMETRIC SERIES

$$a = 15$$

$$r = -0.2$$

$$\begin{aligned} \lim S &= \frac{15}{1.2} \\ &= 12.5 \end{aligned}$$

SEQUENCES AND SERIES

185

WRITE

.12

AS A FRACTION

$$\lim S = 16$$

$$r = \frac{3}{4}$$

$$.121212 = \frac{12}{100} + \frac{12}{10^4} + \frac{12}{10^6}$$

$$\frac{a}{1 - \frac{3}{4}} = 16$$

GEOMETRIC SERIES

$$a = \frac{12}{100} \quad r = \frac{1}{100}$$

$$\lim S = \frac{a}{1-r} = \frac{4}{33}$$

$$a = 4$$

CALCULATOR $4 \div 33 = .12$

$$\lim S = 12$$

$$a = 8$$

A.P.
2, a, b.

G.P.
a, b, 9.

$$\frac{8}{1-r} = 12$$

$$a = \frac{2+b}{2} \quad b^2 = 9a$$

$$8 = 12 - 12r$$

$$b^2 = \frac{18+9b}{2}$$

$$r = \frac{1}{3}$$

$$2b^2 - 9b - 18 = 0 \quad \text{FORMULA}$$

$$b = 6 \quad a = 4 \quad b = -1\frac{1}{2} \quad a = \frac{1}{4}$$

SEQUENCES & SERIES

186.

\$5000 BORROWED @ 18% REDUCIBLE
(INTEREST COMPOUNDED MONTHLY ON BALANCE)
CALCULATE REPAYMENTS P.M. OVER 4 YEARS

BORROWING

INTEREST
ON PRINCIPAL
+ PRINCIPAL

$$5000 \times 1.015^{48}$$

SAVING

INTEREST ON
PAYMENTS
+ PAYMENTS

$$x + (1.015x) + (1.015 \times 1.015x) + (1.015 \times 1.015 \times 1.015x) + \dots$$

(\$x p.m.)

$$x(1 + 1.015 + 1.015^2 + \dots + 1.015^{47})$$

(GEOMETRIC SERIES)

$$a=1, r=1.015, n=48$$

$$\text{FORMULA } S = \frac{a(r^n - 1)}{r - 1}$$

SINCE a IS ALWAYS 1

$$S = \frac{r^n - 1}{r - 1}$$

THE LOAN IS PAID WHEN

UNDERSTAND
ONCE.
DO NOT
LEARN.

BORROWING

EQUALS

SAVING $B = Sx$

LOOK AND
LEARN AS A
PATTERN USING
AVAILABLE DATA

VISUALISE &
ONLY PRACTISE THIS

$$x = 5000 \times 1.015^{48}$$

$$x = \$146.88 \text{ p.m.}$$

$$x \times \frac{0.015}{1.015^{48} - 1}$$

PAY \$7050.24
PRINCIPAL 5000
INTEREST \$2050.24

\$512.56

PER YEAR

DIVIDE BY 50 !

$10\frac{1}{4}\%$ p.a
FLAT
RATE

SEQUENCES & SERIES

187.

\$28000 @ 12% p.a. REDUCIBLE p.m.
 REPAYMENTS \$350. BALANCE AFTER 1 YEAR
 FLAT RATE £ NUMBER OF PAYMENTS

**BALANCE = PRINCIPAL + INTEREST LESS
 PAYMENTS + INTEREST**

Look & LEARN AS A PATTERN
 (VISUALISE) OF AVAILABLE DATA

$$\text{BALANCE} = (28000 \times 1.01^{12}) - \left(350 \times \frac{(1.01^{12} - 1)}{.01} \right)$$

$$= \$ 27, 112.22$$

PAID OFF WHEN

COMPARE PATTERN PAGE 186

$$350 = 28000 \times 1.01^n \times \frac{.01}{1.01^n - 1}$$

i.e. $35000 \times 1.01^n - 35000 = 28000 \times 1.01^n$

$$1.01^n = 5$$

$$n \log 1.01 = \log 5$$

$$n \div 162$$

FLAT RATE: $162 \times 350 - 28000 = \dots \div 13.5 \div 280$
 $= 7.6 \%$

EFFICIENCY: WHAT % IS 2126 OF 28000
 NOT $\times 100$, BUT

$$2126 \div 280$$

SEQUENCES & SERIES

188.

\$6000 @ 8% p.a. INTEREST COMPOUNDED QUARTERLY ON BALANCE. 16 QUARTERLY INSTALMENTS

↓ 2%

QUARTERLY INSTALMENT
SAME PATTERN AGAIN (186)

$$6000 \times 1.02 \times \frac{16 \times .02}{(1.02^{16} - 1)}$$

PRINCIPAL 100%
+ INTEREST 2%
102% = 1.02

\$441.90 $\xrightarrow{0.7553}$

FLAT RATE

(SIMPLE INTEREST)

$\downarrow$ YEARS
 $\times 16 - 6000 = \dots \div 4 \div 60$

4.46%

SUPER ANNUATION INVESTMENT

\$750 . BEGINNING EACH YEAR @ 8% p.a. COMPA 20 YEARS

FIRST \$750 WILL INCREASE TO 750×1.08^{20}
LAST \$750 TO 750×1.08

$$S_{20} = 750 \left(1.08 + 1.08^2 + \dots + 1.08^{20} \right)$$

(GEOMETRIC SERIES: $a = 1.08 = r$)

TOTAL
VALUE

INVESTMENT

$$\frac{750 \times 1.08 (1.08^{20} - 1)}{0.08}$$

VISUALISE
PATTERN

\$37,067

SEQUENCES & SERIES

189.

$$\sum_{n=1}^{40} (2n+5)$$

SIGMA - GREEK
S FOR SUM

MEANS: SUM THE PROGRESSION
FOR WHICH $T_n = 2n+5$
FOR n FROM 1 TO 40

7+9+11+... 85
FIRST LAST

$$S_{40} = 20(7+85) = 1840$$

$$\sum_{n=1}^{\infty \text{ INFINITY}} 5 \times \left(\frac{1}{4}\right)^{n-1}$$

MEANS: INFINITE SERIES $5 + \frac{5}{4} + \frac{5}{16} \dots$
SINCE $r = \frac{1}{4}$, THE LIMITING SUM

$$S = \frac{a}{1-r} = 6\frac{2}{3}$$

S WILL NEVER EXCEED

SEQUENCES & SERIES

190.

AP 26, 23, 20, ...

^{FIND}
 S_n . LEAST n $S_n < 0$

5, 11, 29, ...

$T_n = 3^n + 2$. FIND T_4 & S_n

$$S_n = \frac{n}{2} \{ 52 - 3(n-1) \}$$

$$\frac{n}{2} (55 - 3n)$$

$$T_4 = 3^4 + 2 = 83$$

$$55 - 3n < 0$$

$$3n > 55$$

$$\text{LEAST } n = 19$$

SEQUENCE REWRITTEN AS

$$(3+2), (3^2+2), (3^3+2), \dots$$

$$S_n = \frac{3(3^n - 1)}{2} + 2n$$

$\log 4 + \log 8 + \log 16 + \dots$

^{FIND}

S_{50}

G.P. 9, 6, ...

^{FIND} T_3, T_n

A.P. 1, 4, 7, ... FIND T_{50} & S_{50}

REWRITE

$$2 \log 2 + 3 \log 2 + 4 \log 2 \dots$$

A.P. $d = \log 2$

$$a = 2 \log 2$$

$$r = \frac{2}{3}, T_3 = 4$$

$$T_n = 9 \times \left(\frac{2}{3}\right)^{n-1}$$

$$S_{50} = 25(4 \log 2 + 49 \log 2)$$

$$1325 \log 2$$

$$T_{50} = 1 + 49 \times 3 = 148$$

$$S_{50} = 25(2 + 49 \times 3) = 3725$$

SEQUENCES & SERIES

191.

A.P. $S_3 = 27$

$T_4 + T_5 + T_6 = 63$

A.P. 100, 95, ... FIND S_{100}

CONVERT $.4\bar{7}$

$\frac{3}{2}(2a + 2d) = 27 \quad (1)$

$S_6 = 3(2a + 5d) = 90 \quad (2)$

$d = -5$

$S_{100} = 50(200 - 495)$

$- 14750$

$6a + 15d = 90 \quad (2)$

$6a + 6d = 54 \quad (1)$

$9d = 36$

$d = 4, a = 5$

$100 \times .4\bar{7} = 47.4\bar{7}$

$1 \times .4\bar{7} = .4\bar{7}$

$99 \times .4\bar{7} = 47$

$.4\bar{7} = 47/99$ CHECK $47 \div 99$ ✓

$1 + 3 + 5 + \dots$ n TERMS

$2 + 4 + 6 + \dots$ n TERMS

S_{2000} $1 - 2 + 3 - 4 + 5 \dots$

$\frac{1}{3} - \frac{1}{6} + \frac{1}{12} \dots$ FIND S_{10}

CAN THERE BE A G.P. LIM. $S = \frac{5}{8}$
 $a = 2$

$x^2(2a-x)(2b-x)$ SHOW $\frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ IS A.P.

SUM ODDS ARE SQUARES!

$1+3=4$

$1+3+5=9$

$1+3+5+\dots$ $\uparrow$
 $S_n = \frac{n}{2} \{2 + 2(n-1)\} = n^2$

$2+4+6+\dots$

$S_n = \frac{n}{2} \{4 + 2(n-1)\} = n^2 + n$

$1+3+5+\dots$

GROUP 1

$S_{1000} = 1000^2$

$-2 - 4 - 6 \dots$

GROUP 2

$S_{1000} = \dots$

$S_{2000} = -1000$ GROUPS 1+2

$r = -\frac{1}{2}$
 $S_{10} = \frac{\frac{1}{3} \{1 - (-\frac{1}{2})^{10}\}}{\frac{3}{2}} = \frac{341}{1536}$

$S = \frac{2}{1-r} = \frac{5}{8} \therefore r = -11/5$
NO LIM. SUM. $|r|$ MUST BE < 1

DEFINITION A.P.

$\frac{2}{x} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \therefore x = \frac{2ab}{a+b}$

$x^2 = 4ab - x(2a+2b) + x^2$

$\therefore x = \frac{2ab}{a+b}$ TRUE

SEQUENCES & SERIES

192.

A.P. 50, 43, 36

$$T_n = -27$$

FIND S_n

ANNUAL
\$1000 LOAN. REPAYMENTS
@ 10% ON BALANCE. FIND
AMOUNT OWING BEGINNING
2ND & 3RD YEAR.

$$T_n = 50 - 7(n-1) = -27$$

$$n = 12$$

BEG. 2ND: \$1100 - P OWING

BEG. 3RD: \$1210 - 2.1P OWING

$$S_n = S_{12} = 6(50 - 27) = 138$$

USING ROUTINE PATTERN
$$P = \frac{100 \times 1.1^n}{1.1^n - 1}$$

FIND S_{50}

A.P. $T_1 = 1, T_{26} = 2$

FIND S_{10}

A.P. $T_1 = 2, T_4 = 3, T_3$

CONVERT. 303030...

FIND S_6

G.P. $T_3 = 54, T_6 = 2$

$$T_{26} = 1 + 25d = 2$$

$$d = 1/25$$

$$S_{50} = 25(2 + 49/25) = 99$$

$$2 + 3d = 6 + 6d \therefore d = -4/3$$

$$S_{10} = 5(4 - 12) = -40$$

G.P. .30 + .0030 + ...

$$r = 1/100, a = .30$$

$$S = \frac{a}{1-r} = \frac{.30}{.99}$$

$$.30 = \frac{10}{33}$$

$$100 \times .30 = 30.30$$

$$1 \times .30 = .30$$

$$99 \times .30 = 30$$

$$.30 = \frac{10}{33}$$

$$\left. \begin{array}{l} ar^2 = 54 \\ ar^5 = 2 \end{array} \right\} \begin{array}{l} r^3 = 1/27 \\ r = 1/3, a = 486 \end{array}$$

$$S_6 = 486 \left(1 - \frac{1}{3^6} \right) \times \frac{3}{2} = 728$$

SEQUENCES & SERIES

193.

A.P. $T_1 = 4$
 $T_5 = 4T_3$

YEARLY INVESTMENT
SUPER ANNUATION FROM 1.1.57
 \$3000 @ 9% COMP UNTIL 31.12.84

$$T_3 = 4 + 2d$$

$$T_5 = 4 + 4d$$

$$4 + 4d = 16 + 8d$$

$$d = -3$$

ACCUMULATED VALUE,
LAST TO FIRST:
 $3000 \times 1.09 + \dots + 3000 \times 1.09^{28}$
 $3000 (1.09 + \dots + 1.09^{28})$
 $3000 \left\{ \frac{1.09 (1.09^{28} - 1)}{.09} \right\} =$
 $\$369,406$
 30% TAX ON LUMP SUM
 OVER \$50,000
 NET \$273,584

G.P. $1 + x^2 + x^4 + \dots$
 S_n & lim. sum. $r = \frac{1}{3}$

A.P. $T_1 = 10$
 $T_8 = 8$
 FIND S_{21}

A.P. $T_5 = 14$
 $S_{10} = 165$

$$S_n = \frac{1 - x^{2n}}{1 - x^2}$$

$$10 + 7d = 8$$

$$d = -\frac{2}{7}$$

$$5(2a + 9d) = 165$$

$$a + 4d = 14$$

LIMITING SUM

$$S = \frac{1}{8/9}$$

$$= 9/8$$

$$S_{21} = \frac{21}{2} \left(20 - \frac{40}{7} \right)$$

$$= 150$$

$$2a + 9d = 33$$

$$2a + 8d = 28$$

$$d = 5$$

$$T_1 = a = -6$$

SEQUENCES & SERIES 194.

A.P. $S_7 = 5 T_7$
 $T_6 + T_7 = 40$ FIND S_{10}

FIND THE NUMBER, WHICH,
 WHEN ADDED TO 2, 6 & 13.
 WILL GIVE A G.P

$$\frac{7}{2}(2a + 6d) = 5a + 30d$$

$$\therefore \begin{aligned} 2a - 9d &= 0 \quad (1) \\ 2a + 11d &= 40 \quad (2) \end{aligned}$$

$$\underline{\hspace{1cm}} \quad d = 2$$

$$a = 9$$

$$S_{10} = 180$$

BY DEFINITION

$$(x+6)^2 = (x+2)(x+13)$$

$$3x = 10$$

$$x = 3\frac{1}{3}$$

A.P. $T_1 = 10, T_n = 60$
 $S_n = 3535$

A.P. 12, 17, 22, ...
 FIND T_{25} & S_{25}

A.P. $a = 5, d = 1.2$
 $n = 103$ FIND S_{103}

$$\begin{aligned} T_n &= 10 + d(n-1) = 60 \\ d(n-1) &= 50 \end{aligned}$$

$$S_n = \frac{n}{2} \times 70 = 3535$$

$$n = 101$$

$$d = \frac{1}{2}$$

$$T_{25} = 12 + 24 \times 5 = 132$$

$$S_{25} = \frac{25}{2}(24 + 120) = 1800$$

$$S_{103} = \frac{103}{2}(10 + 102 \times 1.2)$$

$$S_{103} = 6818.6$$

SEQUENCES & SERIES

195

SUPERANNUATION INVESTMENT OF \$800
AT THE BEGINNING OF EACH YEAR FROM 1988. LAST
ONE IN 2017. AMOUNT TO WHICH IT WILL HAVE GROWN
BY 2018 & TOTAL AMOUNT SAVED. COMP. INT 10% p.a.

FIRST AMOUNT WILL HAVE GROWN TO $\$800 \times 1.1^{30}$
\$13,960

LAST AMOUNT WILL HAVE GROWN TO $\$800 \times 1.1$

TOTAL (REVERSED) $800(1.1 + 1.1^2 + \dots + 1.1^{30})$ G.P. $a=1.1$
 $r=1.1$
 $S_{30} = \frac{800 \times 1.1(1.1^{30} - 1)}{.1} = \$144,755$

\$80,000 BORROWED AT 2% COMPOUND P.M.
2 ANNUAL INSTALMENTS OF \$M. AMOUNT
OWING AFTER 1M. AMOUNT OWING AFTER 1Y. ME TOTAL INT.

AMOUNT OWING AFTER 1 MONTH:

$$\$80,000 \times 1.02 = \$81,600$$

AMOUNT OWING AFTER 1 YEAR : $\$ (80,000 \times 1.02^{12} - M)$

AFTER 2 YEARS: $80,000 \times 1.02^{24} = M + M \times 1.02^{12}$

$$M = \frac{80,000 \times 1.02^{24}}{(1 + 1.02^{12})} = \$56,729$$

$\frac{16729}{800}$
$= 20.9\%$
FLAT

TOTAL INTEREST PAID: $2M - 80,000 = \$33,458$ $\uparrow$
REDUCIBLE OR COMPOUNDED MONTHLY MEANS: INTEREST ON PAYMENT

SOLIDS

196.

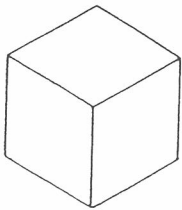
PRISMS

VOLUME

HEIGHT
X
AREA BASE

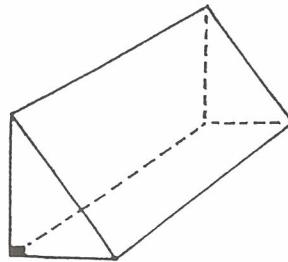
TOP = BASE

THEY
DEFINE THE NAME

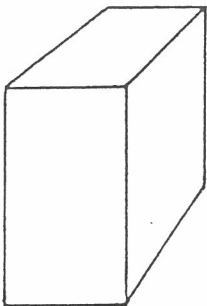


CUBE

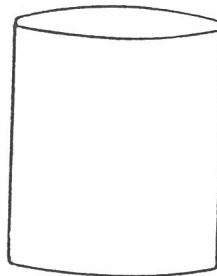
6 FACES
12 EDGES
8 VERTICES



TRIANGULAR PRISM



RECTANGULAR PRISM



CYLINDER

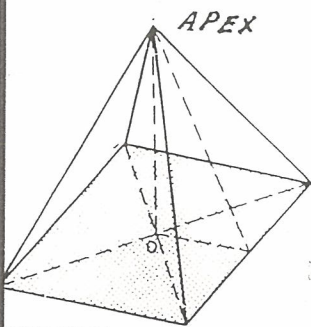
CIRCULAR PRISM

EGYPT PYRAMIDS A M I D S T SAND

VOLUME $\frac{1}{3}$ HEIGHT X AREA BASE

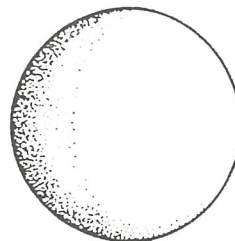
BASE SHAPE

DEFINES THE NAME



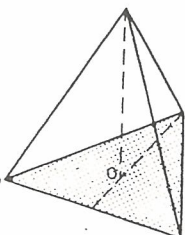
SQUARE PYRAMID

4 SLANT
EDGES
HALF CROSS
SECTION



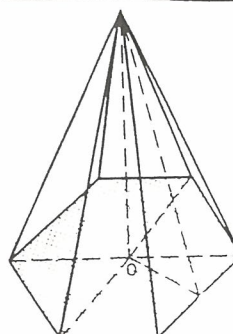
SPHERE

AN INFINITE
NUMBER OF
PYRAMIDS
R = HEIGHT

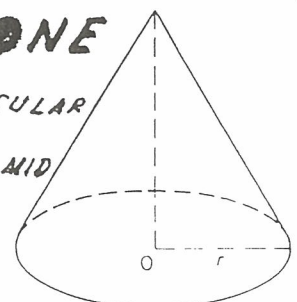


← TRIANGULAR

HEXAGONAL
→



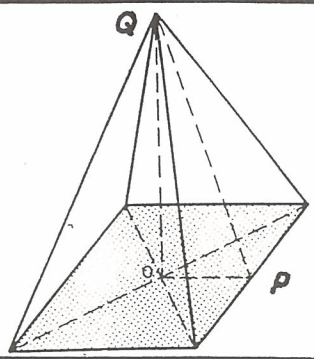
CONE CIRCULAR PYRAMID



SOLIDS

197.

SQUARE PYRAMID



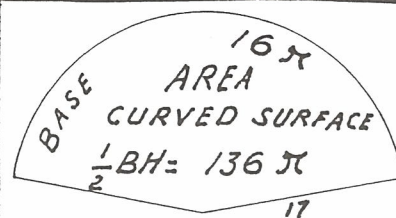
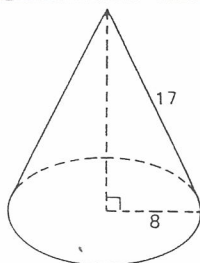
BASE EDGE 6cm
HEIGHT 4cm
QP = 5cm

$$VOLUME = 48 \text{ cm}^3$$

$$SURFACE AREA = 96 \text{ cm}^2$$

CONE

CIRCULAR
PYRAMID



HEIGHT = 15 UNITS (TRIAD)

$$VOLUME = 320 \pi \text{ U}^3. SA = 200 \pi \text{ U}^2$$

700 g COPPER SPHERE. $1 \text{ cm}^3 \text{ COPPER} = 8.95 \text{ g.}$

SURFACE AREA SPHERE =
S.A. CYLINDRICAL CONTAINER
WITHOUT TOP & BASE

$$S.A. = 4R^2\pi$$

2R

$2R\pi$

VOLUME SPHERE = VOLUME SUM OF AN INFINITE
NUMBER OF PYRAMIDS: $\left. \begin{array}{l} \text{TOTAL BASE } 4R^2\pi \\ \text{HEIGHT } R \end{array} \right\} = \frac{4}{3} R^3 \pi \text{ U}^3$

HERE, $\frac{4}{3} R^3 \pi = 700 \div 8.95.$

TO CALCULATE R, DO

$$700 \div 8.95 \times 3 \div 4 \div \pi = \dots \text{ CUBE ROOT}$$

ROUNDED OFF
 $R \div 2.7 \text{ cm}$ (LEAVE ALL DECIMALS

IN THE CALCULATOR)
SQUARE, $\times 4 \times \pi \div 88.4$

$$S.A. \div 88.4 \text{ cm}^2$$

IF NOT,
S.A. = 91.6
 cm^2

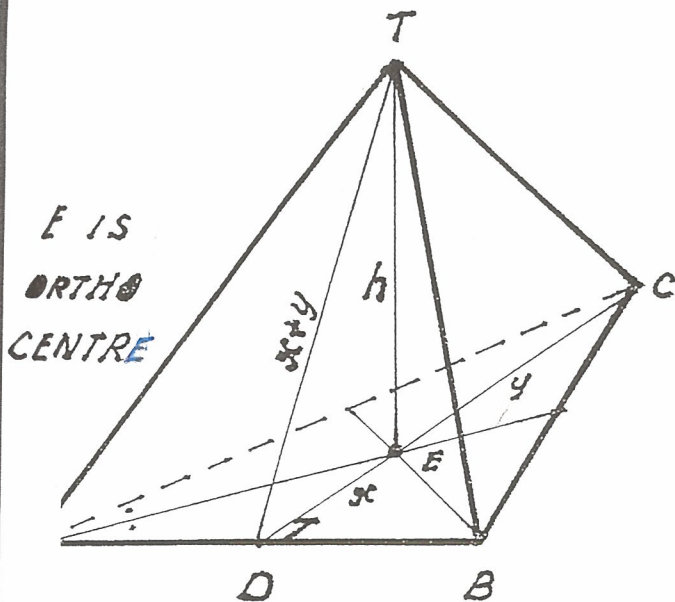
TRIANGULAR PYRAMID

ALL EDGES 6 CM

4 FACES

4 VERTICES

6 EDGES



EQUILATERAL
ALTITUDES ARE ALSO
PERPENDICULAR BISECTORS

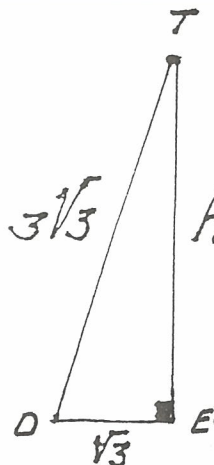
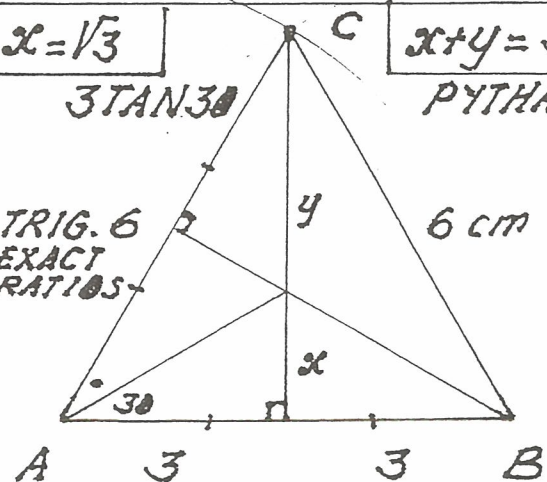
$$x = \sqrt{3}$$

$$x + y = 3\sqrt{3}$$

$$3 \tan 30^\circ$$

PYTHAGORAS

TRIG. 6
EXACT
RATIOS



PYTHAGORAS

$$h = 2\sqrt{6}$$

MENTALLY!

MENTALLY

$$\text{VOLUME} = \frac{1}{3} \times 2\sqrt{6} \times 3 \times 3\sqrt{3}$$

$$18\sqrt{2} \text{ cm}^3$$

SIMILAR SOLIDS

EDGES

$$1:3$$

S.A.

$$1:9$$

VOLUME

$$1:27$$

SURFACE AREA

$$4 \times 3 \times 3\sqrt{3}$$

$$36\sqrt{3} \text{ cm}^2$$

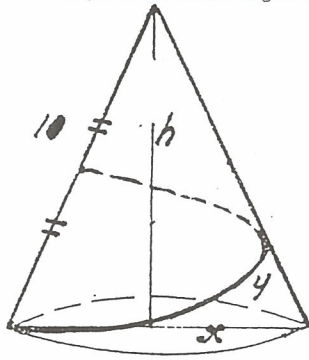
SOLIDS

199

60° SECTOR, $R=10\text{ cm}$

BENT TO FORM A CONE.

PFA SECTOR. V & LENGTH STRING



$$C = 2\pi x = \frac{10\pi}{3}$$

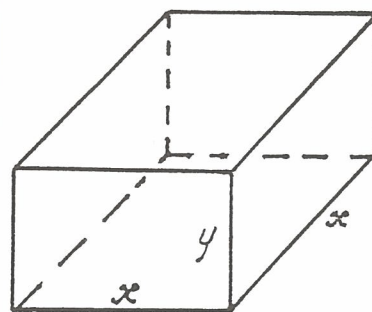
$$x = \frac{5}{3} \text{ cm}$$

$$h = \frac{5\sqrt{35}}{3} \text{ cm}$$

SHEET METAL BOX

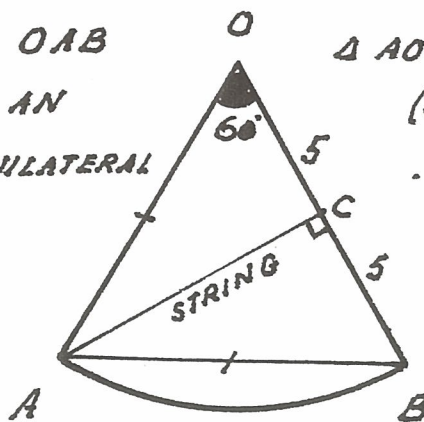
$$500 \text{ cm}^3$$

FIND LEAST AREA



$$y = \frac{500}{x^2}$$

ΔOAB
IS AN
EQUILATERAL



$\Delta AOC \equiv \Delta ABC$
(SSS)

$$\therefore \hat{C} = 90^\circ$$

$$\text{STRING} = \sqrt{75} = 5\sqrt{3} \text{ cm}$$

$$P = \frac{10\pi}{3} + 20 \text{ cm}$$

$$A = \frac{50\pi}{3} \text{ cm}^2$$

$$A = x^2 + 4xy = x^2 + \frac{2000}{x}$$

$$A' = 2x - \frac{2000}{x^2} = 0$$

$$x = 10 \text{ cm. LEAST } A = 300 \text{ cm}^2$$

SQUARE PYRAMID.

$$\text{EDGE } x. V = 750 \text{ cm}^3. H = 8.45 \text{ cm}$$

$$\frac{1}{3} \times 8.45 \times x^2 = 750$$

$$x = 16.32 \text{ cm}$$

$$\text{SPHERE: } V = 5 \text{ cm}^3$$

$$\frac{4}{3} R^3 \pi = 5$$

$$R = 1.06 \dots$$

$$A = 4R^2 \pi = 14.14 \text{ cm}^2$$

VOLUME CONE

$$\frac{5\sqrt{35}}{9} \times \frac{25}{9} \times \pi \text{ cm}^3$$

STRAIGHT LINES

200.

$$y = x + 4$$

LINEAR EQUATION

DOMAIN

x

•

-2

RANGE

y

6

•

3

x & y ARE VARIABLES

ORDERED PAIRS

POINTS

x IS THE INDEPENDENT VARIABLE

IF $x=0$, $y=4$
IF $x=-2$, $y=2$

B

D

y IS THE DEPENDENT VARIABLE

IF $y=6$, $x=2$

A

CARTESIAN COORDINATES

IF $y=0$, $x=-4$

E

x -COORDINATE(S) ABSCISSA(E)

IF $y=3$, $x=-1$

C

y -COORDINATE ORDINATE

- THE FIVE P's -

PATTERN

PAIRS

POINTS

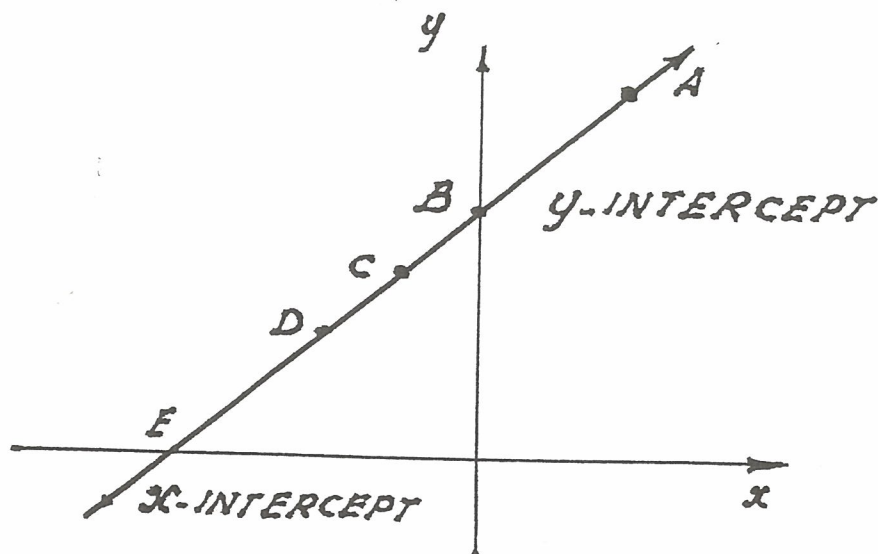
PLOT

PICTURE

YOU NEED

3 PAIRS TO GIVE 3 POINTS

2 AND 1
TO CHECK



STRAIGHT LINES

201

LINEAR EQUATION

GRADIENT-INTERCEPT FORM

- IN THAT ORDER -

$$y = mx + b$$

HEAD

DIRECTION

LEFT(-) RIGHT(+)

SLOPE

HOW STEEP
STEEPER AS m INCREASES

TAIL

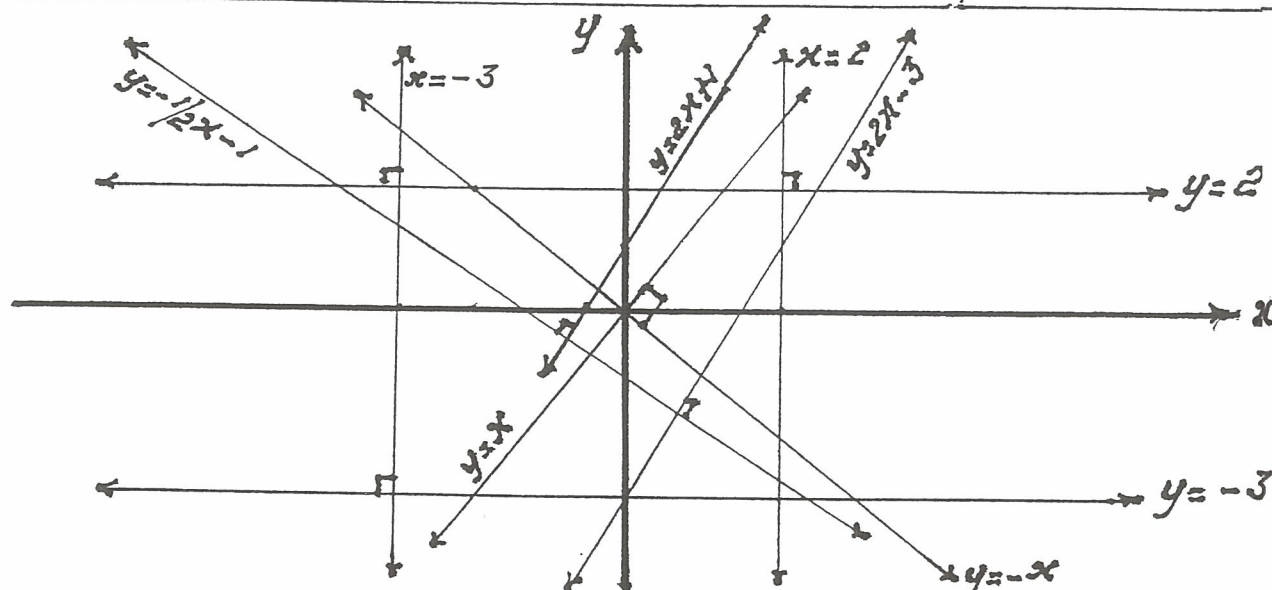
UP,
DOWN OR
THROUGH ORIGIN

VERTICAL		HORIZONTAL		PARALLEL	PERPENDICULAR
$m = \infty$		$m = 0$		EQUAL m	m OPP. RECIPR.
$x = 2$	$x = -3$	$y = 2$	$y = -3$	$y = 2x + 1$ $y = 2x - 3$	$y = 2x + 1$ $y = -\frac{1}{2}x - 1$

IT IS NOT NECESSARY TO USE THE SAME
SCALE FOR THE x - & y COORDINATES.

$$y = x$$

$$y = -x$$



STRAIGHT LINES

202.

FROM
PAIRS

x 0 1 2 3 4

y -1 1 3 5 7

TO

PATTERN
EQUATION

SET OF ORDERED PAIRS

$\{(0, -1), (1, 1), (2, 3), (3, 5), (4, 7)\}$

x y

IN ALPHABETICAL ORDER

USE SIMULTANEOUS EQUATIONS

NO TRIAL & ERROR

SELECT 2 PAIRS: $(2, 3)$ & $(1, 1)$

SUBSTITUTE IN GRADIENT-INTERCEPT FORM

$$y = mx + b$$

$$\begin{array}{r} 3 = 2m + b \\ - \quad 1 = m + b \\ \hline 2 = m \end{array}$$

EQUATION

$$y = 2x - 1$$

STRAIGHT LINES

203.

$$2y - 3x - 6 = 1$$

CHECK
 $P(2,1)$

$$y = -3x - 6$$

GENERAL FORM

$$-3x + 2y - 6 = 1$$

GRADIENT $\downarrow$ NEGATIVE 3
LINE LEANING LEFT

GRADIENT-INTERCEPT FORM

$$y = mx + b$$

3 INV. TAN-72° APPR.

INCLINATION 108°

ANGLE WITH POS. X-AXIS

$$y = \frac{3}{2}x + 3$$

y-INTERCEPT (0, -6)

x-INTERCEPT (-2, 0)

SUB (2,1) $1 \neq 6$

P IS NOT ON THE LINE

$$A(-4, -3), m = 1\frac{1}{2}$$

PROVE $2y = 3x + 6$

$$A(1, 4)$$

LINE $\perp$ $y = -\frac{1}{2}x$ (1)

LINE $\parallel$ $y = 2x + 6$ (2)

VISUALISE

$$y = mx + b$$

SUBSTITUTE MENTALLY!

ONLY WRITE $-3 = -6 + b$
WE DON'T NEED b here.

$$y = \frac{3}{2}x + 3$$

$$2y = 3x + 6 \text{ q.e.d.}$$

(1) $m = -\frac{1}{2}$ (OPP. RECIPROCAL)
 $\therefore$ m LINE THROUGH A IS 2
 $4 = 2 + b$

$$y = 2x + 2$$

(2) $m = 2 = m$ LINE
THROUGH A

$$4 = 2 + b$$

$$y = 2x + 2$$

STRAIGHT LINES

204.

EQUATION LINE, $m=3$
 y -INT = -4. CHECK ORIGIN

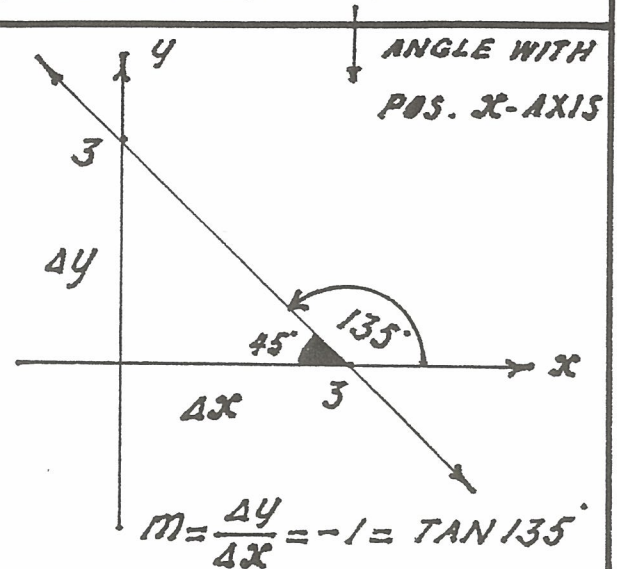
$$y = 3x - 4$$

SUB. (0,0)

$$0 \neq -4$$

THE ORIGIN IS NOT
 ON THE LINE.

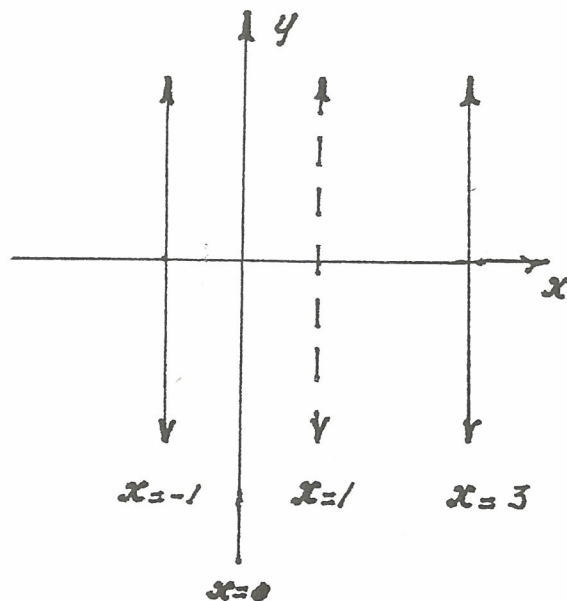
EQUATION LINE, y -INT. = 3
 INCLINATION 135°



$$y = -x + 3$$

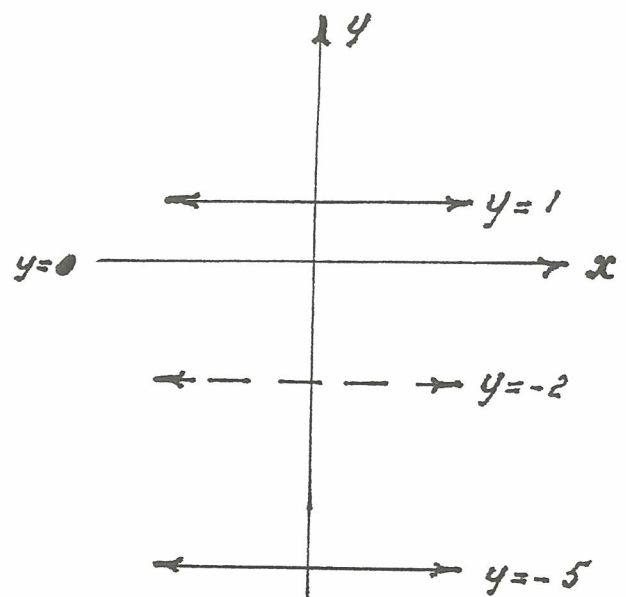
EQUATION LINE $\parallel y$ -AXIS
 MIDWAY $x = -1$ & $x = 3$

$$\bar{x} = (3 - (-1)) \div 2 = 1$$



EQUATION LINE $\parallel x$ -AXIS
 EQUIDISTANT $y = 1$ & $y = -5$

$$\bar{y} = (1 - (-5)) \div 2 = -2$$



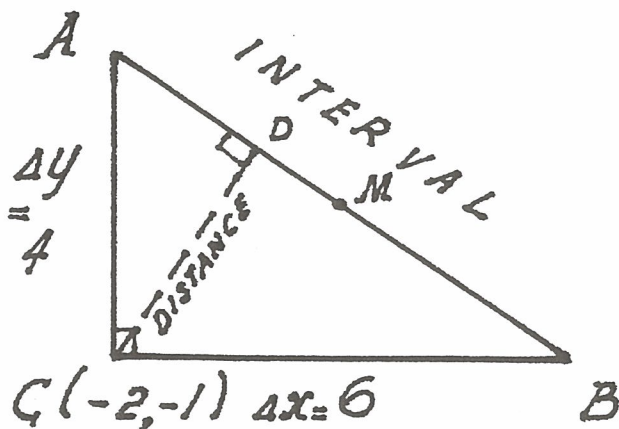
STRAIGHT
LINES

$A(-2, 3)$
2 left, 3 up

$B(4, -1)$
4 right, 1 down

205.

VISUALISE, DRAW WITHOUT AXES,
LABEL AND COMPLETE RIGHT TRIANGLE



EQUATION

THINK (DO NOT WRITE)
GRADIENT, INTERCEPT $y = mx + b$

SUB.

$$m = -\frac{2}{3} \text{ AND } A \text{ OR } B$$

$$3 = \frac{4}{3} + b \quad \left. \begin{array}{l} \text{YOU DON'T} \\ \text{NEED} \\ \text{HERE!} \end{array} \right\} b$$

$$3y = -2x + 5$$

MID POINT

DONE MENTALLY

$$M(\bar{x}, \bar{y})$$

INTERVAL

DO NOT
CALL THAT
DISTANCE

ALWAYS PYTHAGORAS
(NO FORMULA)

TO FIND THE AVERAGE OF
TWO NUMBERS, YOU
DON'T NEED A CALCULATOR
AND CERTAINLY NOT
A FORMULA!

$$AB = 2\sqrt{13} \text{ BINGO!}$$

DISTANCE FROM
A POINT TO A LINE
USE GENERAL FORM $Ax + By + C$

WRITE DIRECTLY

SUB. $C(-2, -1)$ IN

$$2x + 3y - 5 = |-12|$$

$$M(1, 1) \text{ BINGO!}$$

$$\text{DIVIDE BY } \sqrt{2^2 + 3^2}$$

$$\text{DISTANCE} = 3.3 \text{ (CHECK)} \\ \text{CD}$$

DO NOT COMPLICATE SIMPLICITY.

STRAIGHT LINES

206.

DISTANCE FROM $P(-1, 6)$ TO $4x - 3y = -2$

MATCH THE GENERAL FORM OF A STRAIGHT LINE

$$ax + by + c = 0$$

1. SUBSTITUTE $x = -1, y = 6$ $4x - 3y + 2$ $(= 0)$ IGNORE

2. DIVIDE BY $\sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$ -20 } DISTANCE $| -4 | = 4$ UNITS

LEARN THE PROCEDURE

PRACTISE THE DETAILS

A(5, -4)

B(-3, 7)

WHICH POINT IS CLOSER TO
 $5y = -6x + 12$

GENERAL FORM

$$6x + 5y - 12$$

$(= 0)$ IGNORE

SUBSTITUTE

$$x = 5, y = -4$$

SUBSTITUTE

$$x = -3, y = 7$$

-2

5

A IS CLOSER.

A AND B ARE

ON OPPOSITE SIDES

NO NEED TO DIVIDE BY $\sqrt{61}$

STRAIGHT LINES

207.

P MOVES EQUI DISTANT $x \& y$ AXES

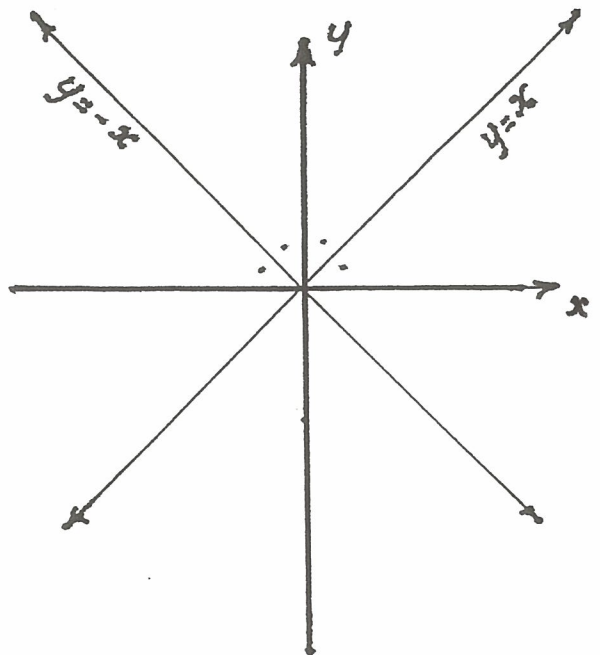
LOCUS OR PATH
(LOCATION)

STRAIGHT LINE

$$y = x$$

OR

$$y = -x$$



P MOVES EQUI DISTANT FROM $A(-1, 6) \& B(3, 2)$

$\Delta x = 4$
 $\Delta y = 4$

LOCUS

RIGHT BISECTOR OF $\overline{AB}$

$$M(\bar{x}, \bar{y}) = (1, 4)$$

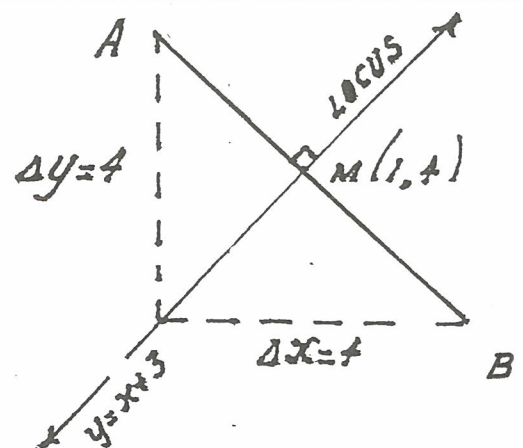
$$m_{AB} = -1$$

$$m_{\text{BISECTOR}} = 1$$

$$t = 1 + b$$

EQUATION
BISECTOR

$$y = x + 3$$



STRAIGHT LINES

208.

EQUATION LINE THROUGH $P(3, -1)$ AND
INTERSECTION Q OF $x + 3y - 4 = 0$ (1) AND
 $3x - 4y + 1 = 0$ (2)

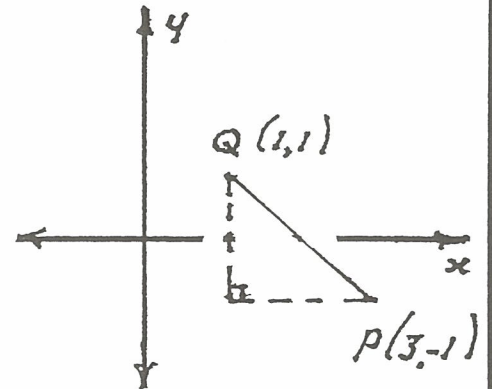
DIRECT METHOD WITH
SIMULTANEOUS EQUATIONS

$$\begin{array}{r} 3x + 9y - 12 = 0 \\ 3x - 4y + 1 = 0 \\ \hline \end{array}$$

$$13y = 13$$

$$y = 1$$

$$x = 1$$



INTERSECTION $Q(1, 1)$

MENTALLY: $m = -1$ LEFT NEG.

$$-1 = -3 + b$$

EQUATION PQ: $y = -x + 2$

K METHOD

IF DIRECT METHOD TOO COMPLICATED

P IS ON THE LINE PQ , SO $x = 3$ AND $y = -1$

MUST SATISFY $(x + 3y - 4) + K(3x - 4y + 1) = 0$

$$-4 + 14K = 0$$

$$K = \frac{2}{7}$$

$$x + y - 2 = 0$$

GENERAL FORM

$$y = -x + 2$$

GRADIENT-INTERCEPT
FORM

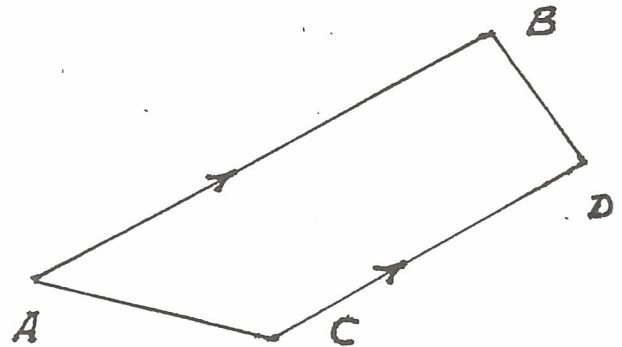
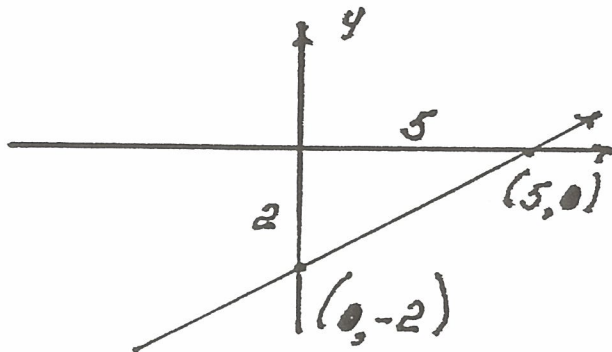
STRAIGHT LINES

209.

x -INTERCEPT
5

y -INTERCEPT
-2

$A(0,2) B(6,5) C(3,1) D(7,3)$
PROVE ABCD IS A TRAPEZIUM



$$m = \frac{2}{5}$$

$$b = -2$$

EQUATION

$$y = \frac{2}{5}x - 2$$

MENTALLY

$$\left. \begin{array}{l} m_{AB} = \frac{1}{2} \\ m_{CD} = \frac{1}{2} \end{array} \right\} \therefore AB \parallel CD$$

$$3x + 5y + 7 = 0$$

1.

$$x + 2y + 2 = 0$$

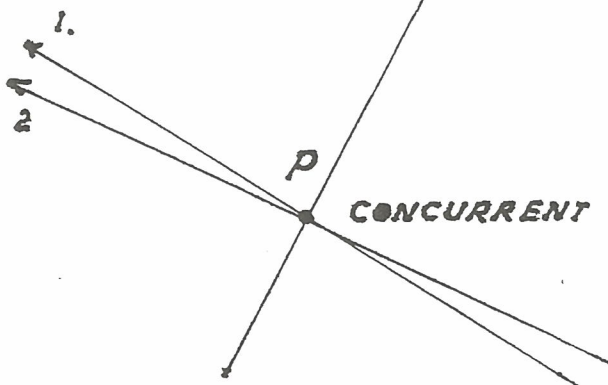
2.

$$2x - y + 9 = 0$$

3.

SHOW
3 LINES
CONCURRENT
TOGETHER RUN

I FIND INTERSECTION
2 LINES (P)
II CHECK IF P
SATISFIES THE 3RD EQUATION



$$\begin{array}{l} 3x + 6y + 6 = 0 \\ 3x + 5y + 7 = 0 \end{array}$$

$$y = 1, x = -4, P(-4, 1)$$

SUB (3) TRUE

LESS you WRITE
MORE you SEE

NOBODY
CAN GIVE A
DEFINITION
IT'S LIKE GOD

NECESSARY WORKING
NECESSARY TO WHOM?

STRAIGHT LINES

210.

SHOW

$$P(3,5) \quad Q(1,1) \quad R(-2,-5)$$

COLLINEAR

1. FIND EQUATION PQ

ON

(THE SAME)

2. CHECK R

LINE

LATIN IS AN EFFICIENT LANGUAGE:
NO FRILLS

MENTALLY (VISUALISE $P \& Q$)

$$\left. \begin{array}{l} P(3,5) \\ Q(1,1) \end{array} \right\} m_{PQ} = 2$$

$$1 = 2 + b$$

EQUATION PQ

$$y = 2x - 1$$

$$\text{SUB. } x = -2, y = -5$$

TRUE

TRIANGLE $P(1,4) \quad Q(-2,-7) \quad R(9,-10)$ PROVE $\angle Q = 90^\circ$

MEANS:

PROVE $PQ \perp QR$ ARE PERPENDICULAR.

IF THEIR GRADIENTS ARE OPPOSITE RECIPROCALLS,

THEY ARE.

WORK WITH AVAILABLE DATA. KEEP THE FORMULA
IN YOUR HEAD. USE THE TOOLS, NOT THE TOOL BOX.
IF YOU WANT TO GO FROM SYDNEY TO BRISBANE
IN A HURRY. DON'T GO TO ADELAIDE FIRST.

MENTALLY

$$m_{PQ} = \frac{11}{3}$$

VISUALISE $P \& Q$

MENTALLY

$$m_{QR} = -\frac{3}{11}$$

VISUALISE $Q \& R$

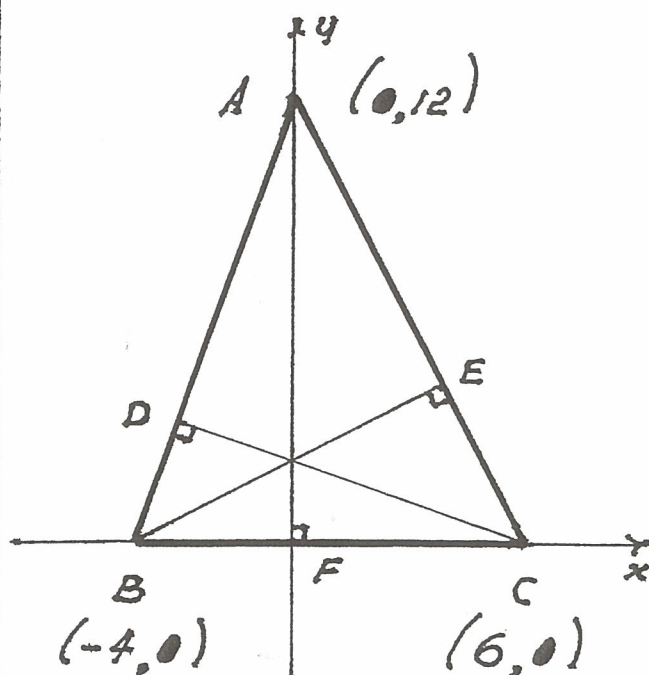
q.e.d.

STRAIGHT LINES

2/1.

TRIANGLE $A(0,12) B(-4,0) C(6,0)$

PROVE CONCURRENT ALTITUDES



$$m_{AC} = -2$$

$$m_{BE} = \frac{1}{2}$$

$$0 = -2 + b$$

EQUATION

$\overline{BE}$

$$y = \frac{1}{2}x + 2$$

$$m_{AB} = 3$$

$$m_{CD} = -\frac{1}{3}$$

$$0 = -2 + b$$

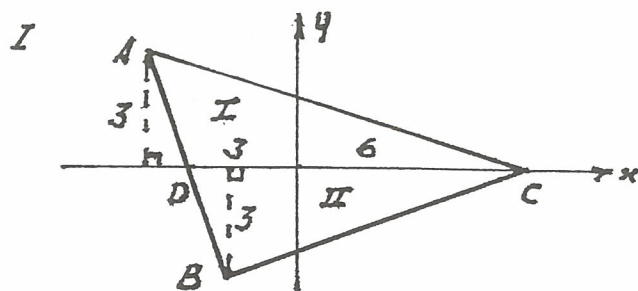
EQUATION

$\overline{CD}$

$$y = -\frac{1}{3}x + 2$$

SAME INTERCEPTS ON AF

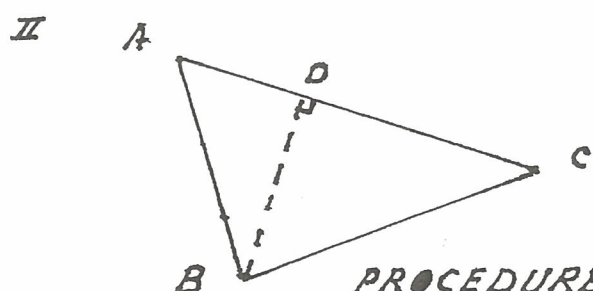
AREA TRIANGLE $A(-4,3) B(-2,-3) C(6,0)$



$$m_{AB} = -3, b = -9, y = 3x - 9$$

$$\text{AT } D, x = -3$$

$$\text{AREA } \triangle ABC = 3 \times 9 = 27 \text{ SQ. UNITS}$$



1. AC, m & EQUATION AC

2. m BD, EQUATION BD

3. INTERSECTION AC & BD

4. BD (PYTHAGORAS)

5. AREA = $\frac{1}{2}$ BD X AC

(SEE 2/4)

STRAIGHT LINES

2/2.

EQUATION OF A LINE

PARALLEL TO $2y - x + 4 = 0$

THROUGH

THE INTERSECTION OF

$$5x - 2y + 3 = 0 \text{ \& } 2x + 6y - 7 = 0$$

THE K -METHOD

WITHOUT CALCULATING
THE INTERSECTION

$$(5x - 2y + 3) + K(2x + 6y - 7) = 0$$

CONVERT TO GENERAL FORM

$$(5 + 2K)x + (6K - 2)y + (3 - 7K) = 0$$

$$m = \frac{5 + 2K}{2 - 6K} = \frac{1}{2}$$

THINK GRADIENT/INTERCEPT FORM!

$$K = -\frac{4}{5}$$

SUBSTITUTE

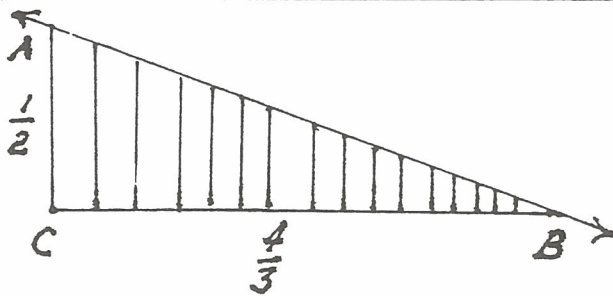
$$17x - 34y + 43 = 0$$

STRAIGHT LINES

2/3.

AREA FORMED BY
 $3x + 8y - 4 = 0$ AXES &

EQUATION LINE
 THROUGH $(2, 7)$
 PARALLEL TO $2x - 3y = 8$



$$y = \frac{2}{3}x - \frac{8}{3}$$

$$A(0, \frac{1}{2}) \quad B(\frac{4}{3}, 0)$$

$$7 = \frac{4}{3} + b$$

$$\text{Area } ABC = \frac{1}{3} \text{ UNITS}^2$$

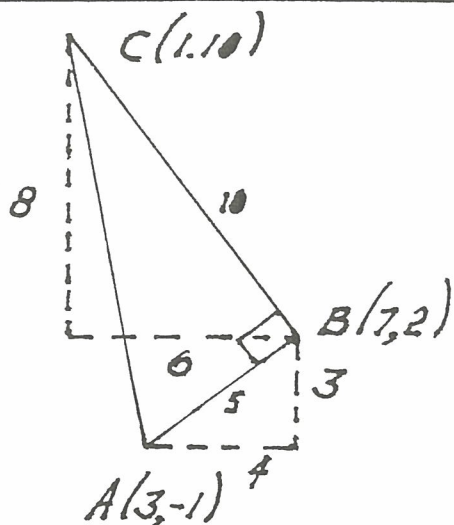
$$y = \frac{2}{3}x + 5\frac{2}{3}$$

AREA TRIANGLE

PARALLELOGRAM ABCD

$$A(3, -1) \quad B(7, 2) \quad C(1, 10)$$

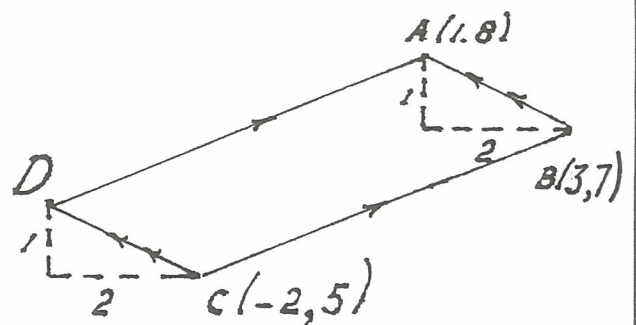
$$A(1, 8) \quad B(3, 7) \quad C(-2, 5)$$



$$m_{AB} = \frac{3}{4}$$

$$m_{BC} = -\frac{4}{3}$$

$AB \perp BC$



COORDINATES $D(-4, 6)$

$$\text{Area Triangle } ABC = 25 \text{ UNITS}^2$$

STRAIGHT LINES

214.

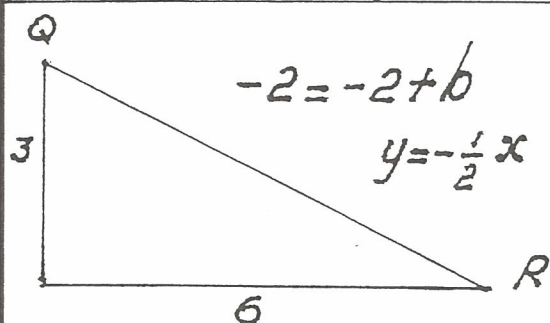
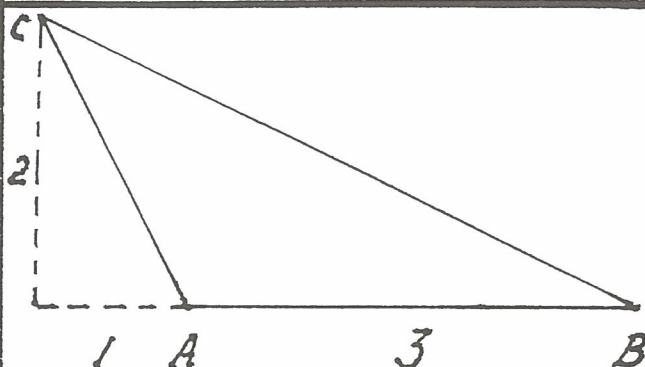
$Q(-2, 1)$ on $9x - 2y + 20 = 0$
 $R(4, -2)$ on $3x + y - 10 = 0$
 EQUATION QR & INTERSECTION LINES

$A(1, 0)$ $B(4, 0)$ $C(0, 2)$

SHOW

$$CB = 2 \times CA$$

$$\begin{array}{r} 9x - 2y + 20 = 0 \\ 6x + 2y - 20 = 0 \\ \hline 15x = 0 \\ x = 0 \end{array} \quad \begin{array}{l} \text{INTERSECTION} \\ (0, 10) \end{array}$$



$$CA = \sqrt{5}$$

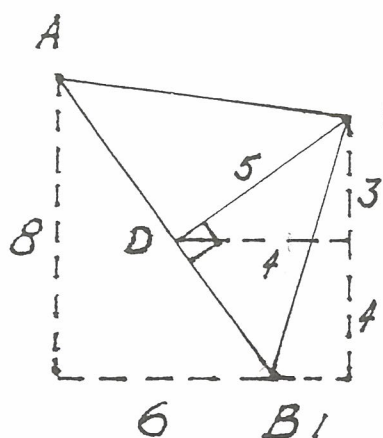
$$CB = 2\sqrt{5}$$

CARTESIAN PLANE (NO PILOTS)

x, y CO-ORDINATES
 $A(-5, 3)$ $B(1, -5)$ $C(2, 2)$
 EQUATION AB & LENGTH

DRAW $CD \perp AB$

FIND CO-ORDINATES &
 AREA TRIANGLE ABC



$$AB = 10$$

$$m_{AB} = -\frac{4}{3}$$

$$-5 = -\frac{4}{3} + b$$

$$\left. \begin{array}{l} m_{CD} = \frac{3}{4} \\ 2 = \frac{6}{4} + b \end{array} \right\} \begin{array}{l} \text{EQUATION CD} \\ y = \frac{3}{4}x + \frac{1}{2} \end{array}$$

GENERAL FORM $3x - 4y + 2 = 0$

ADJUSTED EQUATIONS

$$9x - 12y + 6 = 0$$

$$16x + 12y + 44 = 0$$

$$D(-2, -1)$$

EQUATION AB

$$\text{GENERAL FORM } 4x + 3y + 11 = 0$$

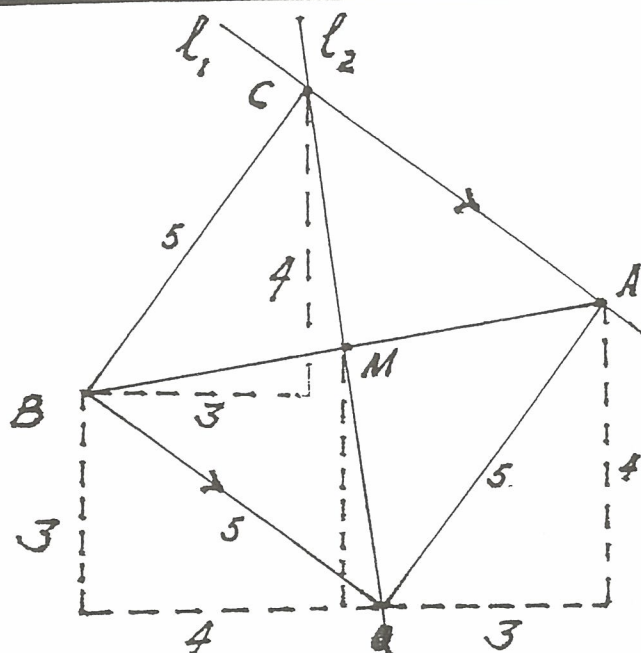
$$\text{AREA } \triangle ABC = 25 \text{ u}^2$$

STRAIGHT LINES

215.

$A(3,4)$ $B(-4,3)$

O IS THE ORIGIN



EQUATION

l_1 THROUGH A PARALLEL TO OB
MIDPOINT AB

$$M(AB) = \left(-\frac{1}{2}, 3\frac{1}{2}\right)$$

$$m l_1 = -\frac{3}{4}$$

$$4 = -\frac{9}{4} + b$$

EQUATION l_1

$$3x + 4y - 25 = 0$$

EQUATION l_2

SHOW $BC \parallel OA$

THROUGH OM

MEETING l_1 AT C

$$x - 2y - 1 = 0 \quad (1)$$

$$2x + y + 18 = 0 \quad (2)$$

DISTANCE $P(8,4)$ TO (1),
INTERSECTION Q. $\hat{P}Q$

$$m l_2 = -7$$

EQUATION l_2 IS $7x + y = 0$

ADJUSTED EQUATIONS l_1 & l_2

$$28x + 4y = 0$$

$$3x + 4y = 25$$

$$C(-1, 7)$$

$$\text{DISTANCE} = \frac{8-8-1}{\sqrt{5}} = \frac{-1}{\sqrt{5}} \text{ UNITS}$$

ADJUSTED EQUATIONS

$$4x + 2y + 36 = 0$$

$$+ \frac{x - 2y - 1 = 0}{}$$

$$Q(-7, -4)$$

$$m BC = \frac{4}{3}$$

$\therefore BC \parallel OA$

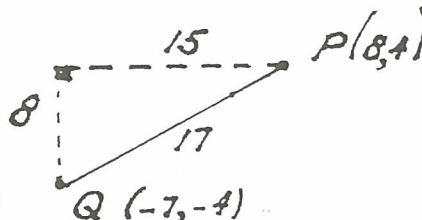
$$OB = AC = BC = OA = 5$$

$$OB \parallel AC, BC \parallel OA$$

$$m OB = -\frac{3}{4}$$

$$m OA = \frac{4}{3}$$

$\hat{O} = 90^\circ$
 $ABCO$ IS A SQUARE



TRIADS

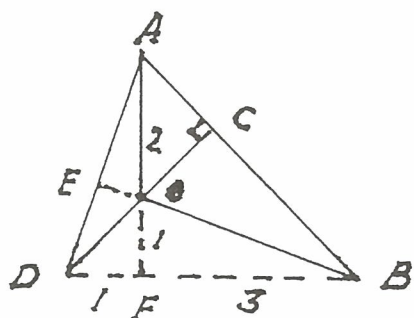
3. 4. 5.
6. 8. 10. ETC.
5. 12. 13.
7. 24. 25.
8. 15. 17.
9. 40. 41.

STRAIGHT LINES

216.

$O(0,0)$ $A(0,2)$ $B(3,-1)$
 $OC \perp AB$

AREA TRIANGLE OAB
 CO-ORDINATES C & D
 PROVE $AD \perp BE$



$$\text{AREA } OAB = 3 \text{ UNITS}^2$$

INTERSECTION

$$y = x$$

$$\& y = -1$$

$$D(-1, -1)$$

$$m_{AB} = -1$$

$$b = 2$$

EQUATION AB

$$y = -x + 2$$

$$m_{OB} = -\frac{1}{3}$$

$$BE \perp AD$$

$$m_{OC} = 1$$

EQUATION OC $y = x$

$$m_{AD} = 3$$

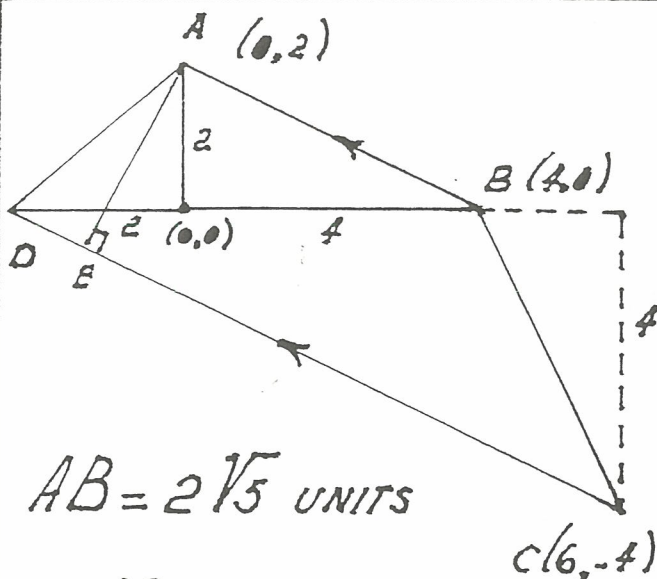
$$OC = \sqrt{2} \text{ (DIST.)}$$

$$x = -x + 2 \quad C(1, 1)$$

3 ALTITUDES ARE CONCURRENT

$A(0,2)$ $B(4,0)$ $C(6,-4)$
 $AB \parallel CD$
 AB & GRADIENT

SHOW
 EQUATION CD: $x + 2y + 2 = 0$
 CO-ORDINATES D, AE &
 AREA TRAPEZIUM ABCD



$$AB = 2\sqrt{5} \text{ UNITS}$$

$$m_{AB} = -\frac{1}{2}$$

EQUATION CD

$$m_{CD} = -\frac{1}{2}$$

$$-4 = -3 + b$$

$$y = -\frac{1}{2}x - 1$$

$$\text{GEN. FORM } x + 2y + 2 = 0$$

$$\text{SUB. } y = 0 \therefore D(-2, 0)$$

$$AE = \frac{6}{\sqrt{5}} \text{ UNITS}$$

AREA TRAPEZIUM:

$$\text{AREA } CBD + \text{AREA } ABD$$

$$12 + 6 = 18 \text{ U}^2$$

$$\text{OR } \frac{3}{\sqrt{5}} (2\sqrt{5} + 4\sqrt{5}) = 18 \text{ U}^2$$

STRAIGHT LINES

217

$Q(1,3) \quad P(3,-2)$

$4x+5y-2=0 (k), \text{ is } P \text{ on } k ?$

$x-3y+8=0 (l), \text{ is } Q \text{ on } l ?$

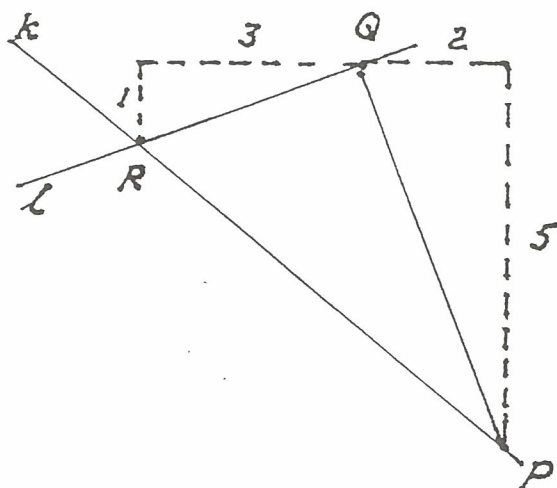
CO-ORDINATES

INTERSECTION $k \cap l (R)$

DISTANCE $P \text{ to } l$

AREA ΔPQR

SUB. $P \& Q$ IN $k \& l$. TRUE



ADJUSTED EQUATIONS

$4x+5y-2=0$

$4x-12y+32=0 \quad R(-2,2)$

DISTANCE $P \text{ to } l = \frac{3+6+8}{\sqrt{10}} = \frac{17}{\sqrt{10}}$

$QR = \sqrt{10}$

AREA $PQR = \frac{1}{2} \times \sqrt{10} \times \frac{17}{\sqrt{10}} = 8\frac{1}{2} \text{ UNITS}^2$

$l) 2x-3y=2. \quad x \text{ INT. } Q.$

$P(4,2) \text{ on } l. \quad k \perp l. \text{ EQUATION } k$

$PR=1 \text{ UNIT. FIND } PQ, \text{ AREA } PQR$

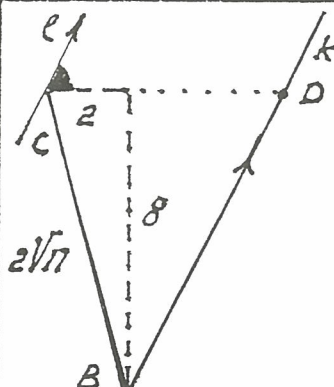
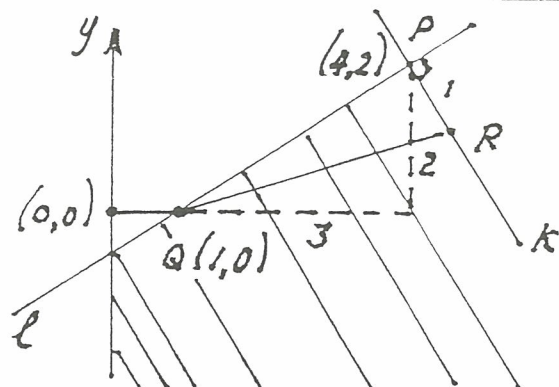
REGION $x > 0, 2x-3y > 2$

$l \text{ THROUGH } (-1,2): y=2x+4$

$k \text{ THROUGH } B(1,-6) \parallel l$

FIND BC , SLOPE ANGLE l

EQ (k) , D & AREA BCD



$m_k = -\frac{3}{2}$ EQUATION k PQ
 $2 = -6 + b$ $y = -\frac{3}{2}x + 8 = \sqrt{13}$

$m_l = 2$, INCLINATION 63°
 2 INV. TAN

AREA $PQR = \frac{1}{2} \sqrt{13} \text{ UNITS}^2$

$B \text{ on } k \parallel l \therefore -6 = 2 + b$

TEST $(0,0)$ FALSE REGION AS SHADED

EQ. $k: y = 2x - 8$. SUB. $y = 2$
 $D(5,2). CD = 6. A = 24 \text{ U}^2$

STRAIGHT LINES

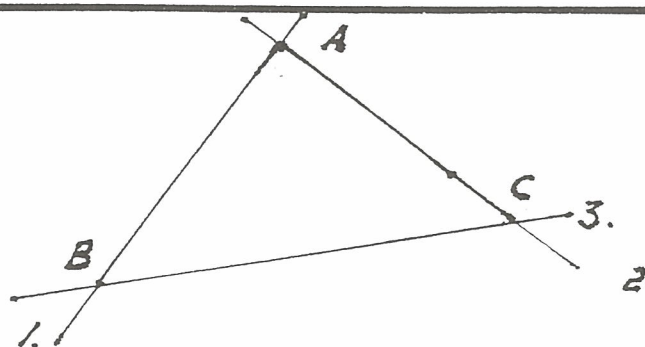
218

TRIANGLE SIDES
ON $2x - y = 0$ (1)

$$x + 2y = 5 \quad (2)$$

$$x - 3y = 20 \quad (3)$$

PROVE
ISOSCELES RIGHT TRIANGLE

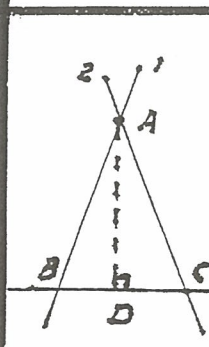


TRIANGLE SIDES
ON $3x - y = 6$ (1)

$$2x + y = 14 \quad (2)$$

$$y = 0 \quad (3)$$

AREA



$$A(4, 6)$$

$$B(2, 0)$$

$$C(7, 0)$$

AREA

$$15 \text{ U}^2$$

GRADIENT/INTERCEPT FORM

$$(1) y = 2x, m = 2$$

$$(2) y = -\frac{1}{2}x + 2\frac{1}{2}, m = -\frac{1}{2}$$

$$\therefore \hat{A} = 90^\circ$$

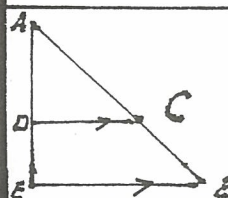
$$A(-2, 3), B(3, -7)$$

C DIVIDES AB RATIO 3:2

$$2x - y = 0 \quad (1)$$

$$2x + 4y = 10 \quad (2)$$

$$A(1, 2)$$



$$\frac{AD}{DB} = \frac{3}{5}, AD = 6$$

$$\frac{DC}{CB} = \frac{3}{5}, DC = 3$$

$$C(1, -3)$$

$$x + 2y = 5 \quad (2)$$

$$x - 3y = 20 \quad (3)$$

$$C(11, -3)$$

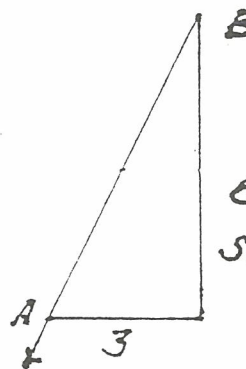
$$(2, 7), (5, 13), (-4, -5)$$

SHOW POINTS ARE COLLINEAR

$$2x - y = 0 \quad (1)$$

$$2x - 6y = 40 \quad (3)$$

$$B(-4, -8)$$



$$m_{AB} = 2$$

$$13 = 10 + 3$$

$$y = 2x + 3$$

SUB. C

$$-5 = -8 + 3$$

TRUE

$$\Delta x_{AB} = 5, \Delta y = 10$$

$$\Delta x_{AC} = 10, \Delta y = 5$$

$$AB = 5\sqrt{5}$$

$$AC = 5\sqrt{5}$$

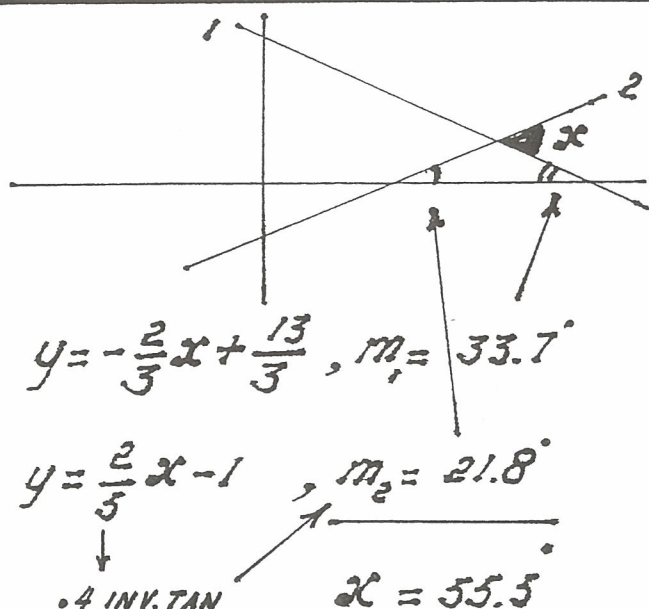
STRAIGHT LINES

219

ACUTE ANGLE
BETWEEN

(1) $2x + 3y = 13$

(2) $2x - 5y = 5$



DISTANCE

$(2, -6)$

$4x - 3y + 9 = 0$

SUBSTITUTE

$$\frac{35}{\sqrt{16+9}}$$

7 UNITS

$3x + 2y = 12 \perp 2x + 4y = 6$
INTERSECTION AT $(2, 3)$

$P(3, 4) \quad Q(x, 0)$

$PQ \parallel y = 2x + 9$

MIDPOINT PQ

$y = -\frac{3}{2}x + 6$

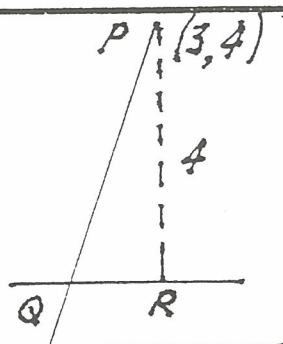
$-\frac{3}{2} = \frac{a}{2}$

$y = -\frac{a}{2}x + \frac{b}{2}$

$a = -3$

SUB. $(2, 3)$
 $a = -3$

$b = -5$



$m_{PQ} = 2$

$\therefore QR = 2$

$Q(1, 0)$

$M(2, 2)$

$4 = 6 + b$

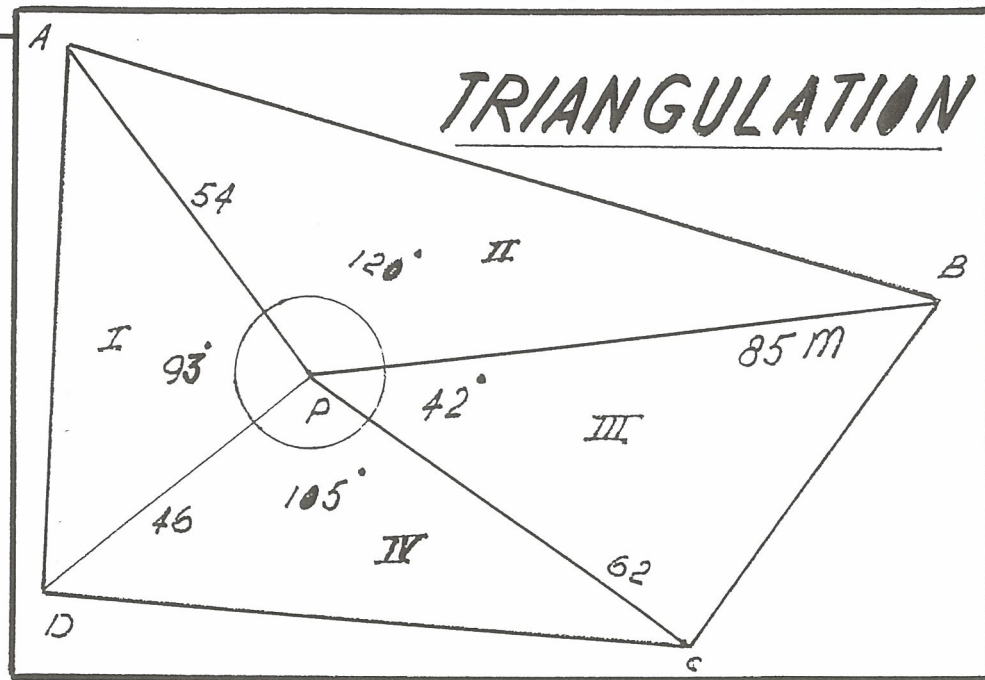
$y = 2x - 2$

EQUATION PQ

PLANE TABLE RADIAL SURVEY

220.

USING A SIGHTING RULE (ALIDADE), A PRISMATIC COMPASS
A SEXTANT OR A THEODOLITE



1. DRAW LINES FROM P IN THE DIRECTION OF A, B, C & D .
2. MEASURE THE ANGLES BETWEEN THESE LINES
3. MEASURE THE ACTUAL DISTANCES PA, PB, PC & PD ,
4. USE A SUITABLE SCALE TO DRAW THE DIAGRAM.

TOTAL AREA

6368 m^2

PERIMETER

$120 + 56 + 86 + 73 =$

335 m

USE SCALE DRAWING COS RULE

I

$.5 \times 54 \times 46 \times \sin 93$

II

$.5 \times 54 \times 85 \times \sin 120$

III

$.5 \times 85 \times 62 \times \sin 42$

IV

$.5 \times 62 \times 46 \times \sin 105$

SCALE

$1:1000$

MEANS 1 CM

REPRESENTS

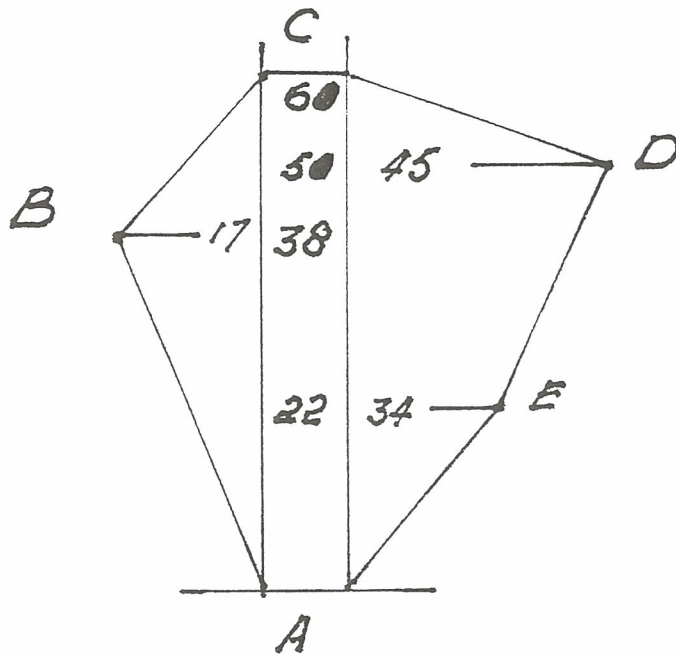
1000 CM

OR 10 m

OVER

LOOK (SEE)

ACROSS



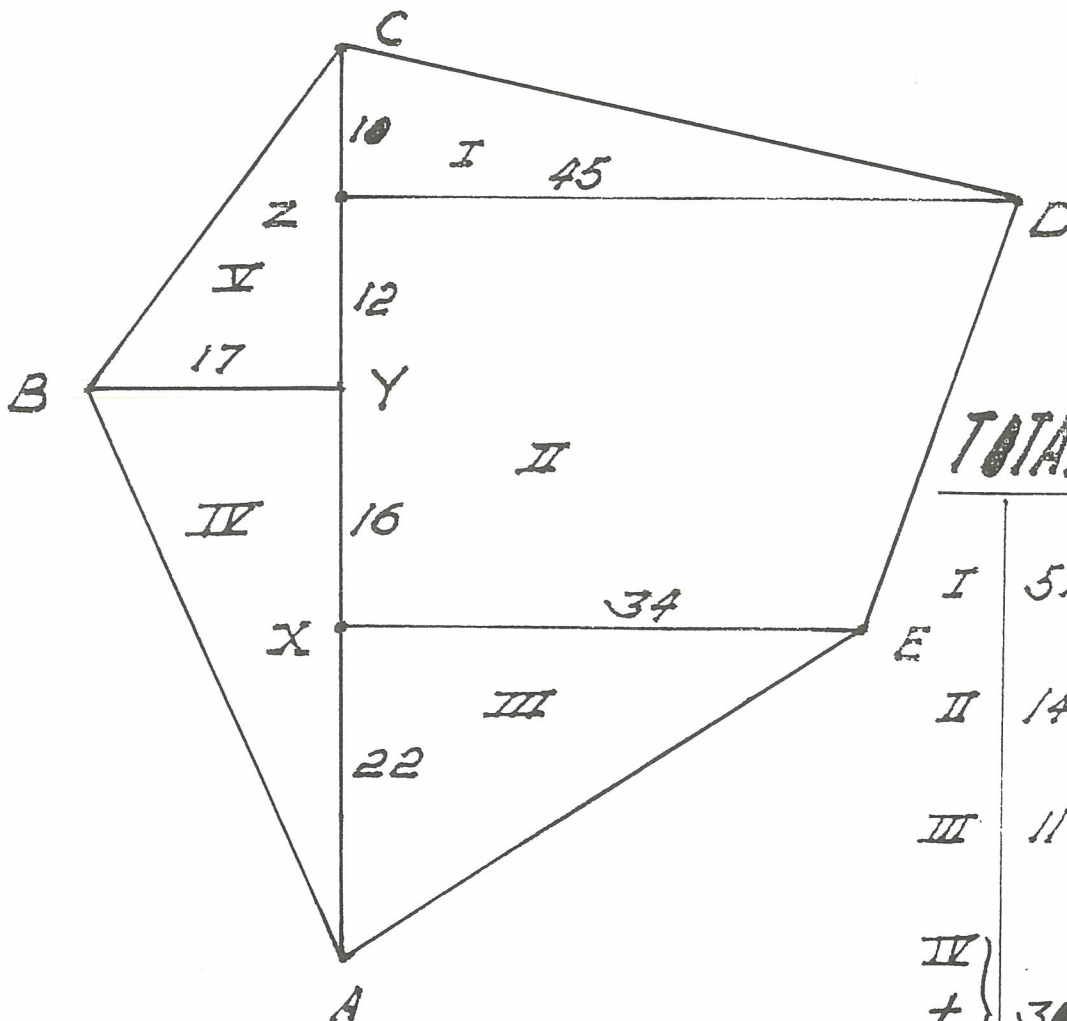
FIELD BOOK

SKETCH

DISTANCES IN m



READ FROM
THE
BOTTOM UPWARDS



TOTAL AREA

I	5X45
II	14X79
III	11X34
IV	30X17
+	
V	

2215 m²

TANGENT & NORMAL

222.

$$y = x^3 - 9x^2 + 20x - 8 \text{ AT } P(1, 4)$$

$$\text{AT WHAT POINTS } // y = -4x + 3$$

$$y' = m \text{ TANGENT} = 3x^2 - 18x + 20 = 5 \text{ AT } P$$

$$4 = 5 + b$$

$$y = 5x - 1$$

TANGENT PARALLEL TO $y = -4x + 3$, WHEN $y' = -4$

$$3x^2 - 18x + 20 = -4$$

$$x^2 - 6x + 8 = 0$$

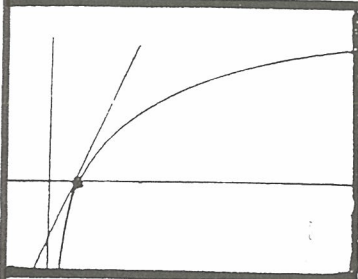
$$(x-4)(x-2) = 0$$

THIS OCCURS

AT

(4, -8) & (2, 4)

$$y = \log_e x \text{ AT } P(1, 0)$$



$$y' = m \text{ TANGENT} = \frac{1}{x} = 1 \text{ AT } P$$

$$0 = 1 + b$$

$$y = x - 1$$

TANGENT & NORMAL

223.

SHOW $3x + 4y + 10 = 0$ IS TANGENT
TO CIRCLE $R=2$, CENTRE ORIGIN

SHOW
MEANS: $(0,0)$ IS 2 UNITS FROM THE LINE

DISTANCE: LEARN WHAT TO DO.

YOU DON'T HAVE TO SHOW THE TEACHER: HE KNOWS!
YOUR MECHANIC GIVES YOU THE BILL FOR THE REPAIRS;
HE DOESN'T WRITE DOWN HOW HE DID THEM.

SUB. $(0,0)$ & DIVIDE BY
 $\sqrt{A^2+B^2}$

$$\frac{10}{5} = 2 \quad \text{q.e.d.}$$

$$y = ax^2 + bx + c$$

$$y=1 \text{ \& \#39; } \frac{dy}{dx} = 1 \text{ WHEN } x=1$$

SHOW $a=c$

WHEN $x=1$,

$$y' = 2a + b = 1$$

$$y = a + b + c = 1$$

$$a - c = 0$$

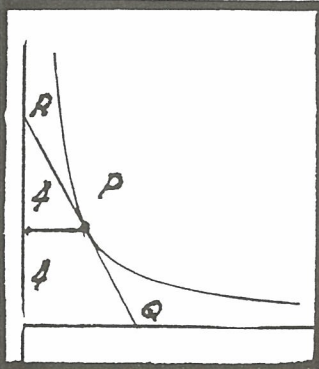
$$a = c \quad \text{q.e.d.}$$

EQUATIONS TANGENT & NORMAL

224.

TANGENT TO $y = \frac{12}{x}$ AT $P(3, 4)$

x-INT Q, y-INT R. IN WHAT RATIO DOES P DIVIDE QR.



$$m_{TAN} = y' = \frac{-12}{x^2} = -\frac{4}{3} \text{ AT } P$$

$$4 = -4 + b$$

EQUATION TANGENT

$$y = -\frac{4}{3}x + 8$$

$Q(6, 0)$

$R(0, 8)$

P IS MIDPOINT QR

$RP:PQ = 1:1$

TANGENTS TO

$y = \log_e x$ & $y = \log_e 2x$

ARE PARALLEL WHEN $x=1$.

$$y = \log_e x$$

$$y = \log_e 2x$$

FUNCTION OF A FUNCTION
DER. PART X DER. WHOLE

$$m_{TAN} = y' = \frac{1}{x} = 1 \text{ WHEN } x=1$$

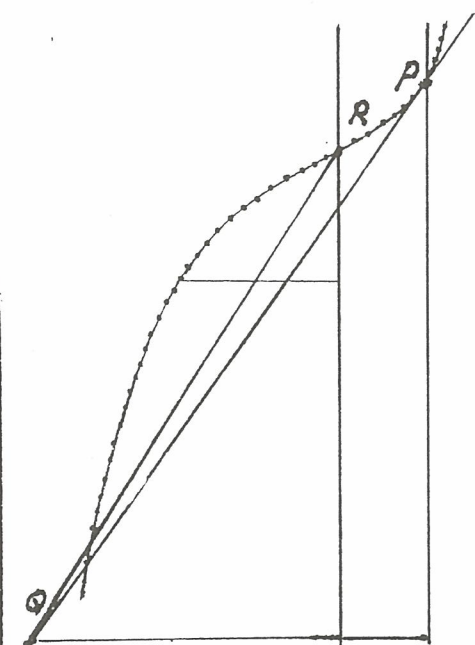
$$m_{TAN} = y' = \frac{2}{2x} = \frac{1}{x} = 1 \text{ WHEN } x=1$$

SINCE THE TANGENTS HAVE THE SAME GRADIENTS,
THEY ARE PARALLEL.

TANGENT & NORMAL

225.

$y = x^3 + 5 \cap x = 1$ (P). TANGENT AT P. CHECK
 $Q(-5, -12)$ ON TANGENT. Y-INT. CURVE IS R
 FIND EQUATION QR. ANGLE BETWEEN QR & QP (α)



$$P(1, 6). \quad m_{\text{NORMAL}} = -\frac{1}{3}$$

$$m_{\text{TAN}} = y' = 3x^2 = 3$$

$$6 = 3 + b \quad \text{EQ. TAN: } y = 3x + 3$$

$$y = 3x + 3$$

$$\text{SUB } Q(-5, -12): -12 = -15 + 3 \quad \checkmark$$

$$R(0, 5)$$

$$m_{\text{QR}} = \frac{17}{5}, \quad b = 5$$

$$\text{EQ. QR: } y = \frac{17}{5}x + 5$$

$$\alpha = \text{INV TAN } \frac{17}{5} - \text{INV TAN } 3 = 2^\circ$$

$$y = x^{1/2}$$

TANGENT &
 AT $P(9, 3)$

NORMAL

$$m_{\text{TAN}} = y' = \frac{1}{2\sqrt{x}} = \frac{1}{6} \quad \text{AT } P$$

$$3 = \frac{9}{6} + b$$

$$\text{EQ. TAN: } y = \frac{1}{6}x + \frac{3}{2}$$

$$m_{\text{NORM.}} = -6$$

$$3 = -54 + b$$

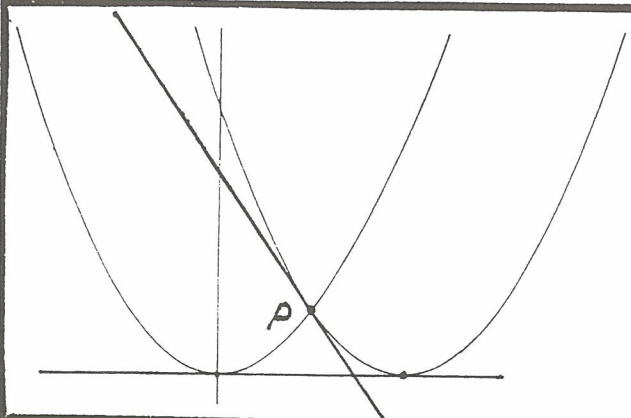
$$y = -6x + 57$$

EQ. NORM

TANGENT & NORMAL

226.

SHOW $P(2,1)$ IS THE ONLY INTERSECTION OF
 $4y = x^2$ & $4y = (x-4)^2$. EQ. TAN P & INT Q WITH $4y = x^2$



2 PARABOLAS: SAME SIZE
 FOCAL LENGTH (Q) = 1
 VERTICES $(0,0)$ & $(4,0)$
 (STANDARD PARABOLA: $x^2 = 4ay$)

AT INTERSECTION $x^2 = (x-4)^2$
 $x^2 = x^2 - 8x + 16$ WHICH IS AT $(2,1)$

$$4y = (x-4)^2$$

$$y = \frac{1}{4}x^2 - 2x + 4$$

$$y' = \frac{1}{2}x - 2 = -1 \text{ AT } P$$

$$1 = -2 + b$$

$$\text{EQ. TAN: } y = -x + 3$$

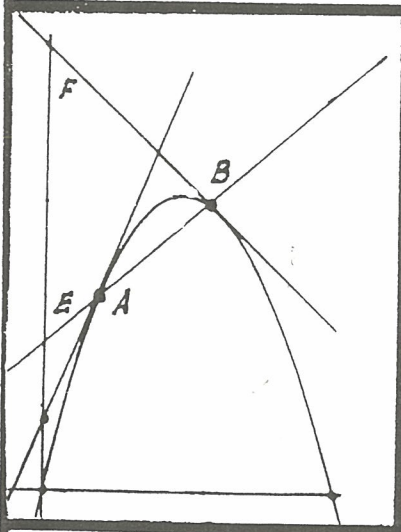
$$4y = -4x + 12$$

$$\text{AT Q, } 0 = x^2 + 4x - 12$$

$$(x+6)(x-2) = 0$$

$$Q(-6, 9)$$

TANGENTS TO $y = 5x - x^2$ AT INTERSECTIONS
 A & B WITH $y = x + 3$. Y-INT. OF TANGENTS
 E & F . FIND EF .



$$\text{AT INTERSECTIONS, } x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$A(1, 4) \text{ \& } B(3, 6)$$

$$y = -x^2 + 5x$$

$$6 = -3 + b$$

$$y' = m \text{ TAN } B = -1$$

$$\text{EQ. TAN } B: y = -x + 9$$

$$y = -x^2 + 5x$$

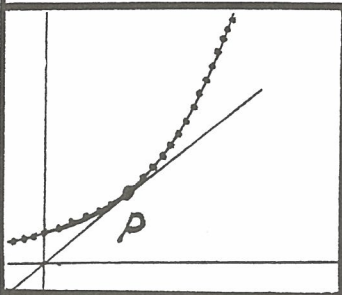
$$4 = 3 + b$$

$$y' = -2x + 5 = 3 \text{ AT } A$$

$$\text{EQ. TAN. } A: y = 3x + 1$$

EF , THE DIFFERENCE OF THE Y-INTERCEPTS, IS 8 UNITS

$$y = mx \text{ IS TANGENT TO } y = e^{3x}$$



$$\text{AT } P, mx = e^{3x}$$

$$m = y' = 3e^{3x}$$

$$\frac{e^{3x}}{x} = 3e^{3x}$$

$$x = \frac{1}{3}$$

$$\underline{m = 3e}$$

$$\text{TANGENT TO } y = x \sin x \text{ AT } P(\pi, 0)$$

REMEMBER

PRODUCT RULE PATTERN: $y'y + y'y'$

SIN = POS COS

$$m = m_{\text{TANGENT}} = y'$$

$$= \sin x + x \cos x = -\pi \text{ AT } P$$

MENTAL SUBSTITUTION

$$0 = -\pi^2 + b$$

$$\underline{\text{EQUATION TANGENT } y = -\pi x + \pi^2}$$

$$\text{TANGENT TO } y = xe^x \text{ AT } P(1, e)$$

$$m_{\text{TANGENT}} = y' = e^x + xe^x = 2e$$

AT P

MENTAL SUBSTITUTION

$$e = 2e + b$$

EQUATION TANGENT

$$y = 2ex - e$$

$$m_{\text{NORMAL}} = -\frac{1}{2e}$$

$$y = 3x + e^{-x} \text{ TANGENT AT } (0, 4)$$

F.O.F.

DER. PART X DER. WHOLE

$$m_{\text{TANGENT}} = y' = 3 - e^{-x} = 2$$

$$\text{AT } (0, 1)$$

MENTAL SUBSTITUTION

$$1 = b$$

EQUATION TANGENT $y = 2x + 1$

$$m_{\text{NORMAL}} = -\frac{1}{2}, b \text{ IS ALSO } 1$$

$$\underline{\text{EQUATION } y = -\frac{1}{2}x + 1}$$

$$\text{To } y = x^2 + \frac{5}{x} - 2 \text{ AT } P(1,4)$$

$$y' = 2x - \frac{5}{x^2} = -3 \text{ AT } P$$

$$4 = -3 + b$$

EQUATION TANGENT

$$y = -3x + 7$$

$$m \text{ NORMAL} = \frac{1}{3} \text{ AT } P$$

$$4 = \frac{1}{3} + b \quad \text{EQ. } y = \frac{1}{3}x + \frac{11}{3}$$

$$\text{To } y = 3 \ln x + 2 \text{ AT } P(1,4)$$

$$m \text{ TANGENT} = y' = \frac{3}{x} = 3 \text{ AT } P(1,2)$$

$$(\ln 1 = 0, \text{ BECAUSE } e^0 = 1)$$

$$2 = 3 + b$$

EQUATION TANGENT $y = 3x - 1$

$$m \text{ NORMAL} = -\frac{1}{3} \text{ AT } P$$

$$2 = -\frac{1}{3} + b$$

$$\text{EQ. } y = -\frac{1}{3}x + \frac{7}{3}$$

$$\text{CURVE THROUGH } (2,3)$$

$$\text{GRADIENT FUNCTION } 3x^2 - 2x + 1$$

MEANS $y' = 3x^2 - 2x + 1$

INTEGRATE

$$y = x^3 - x^2 + x + C$$

SUB(2,3) TO FIND C

EQUATION CUBIC CURVE

$$y = x^3 - x^2 + x - 3$$

$$\text{To } y = x^2 + x \text{ AT } P(1,4)$$

$$y' = 2x + 1 = 3 \text{ AT } P(1,2)$$

$$2 = 3 + b$$

EQUATION TANGENT

$$y = 3x - 1$$

$$\text{EQ. NORMAL } y = -\frac{1}{3}x + \frac{7}{3}$$

TRIANGLES

229.

S.S.S

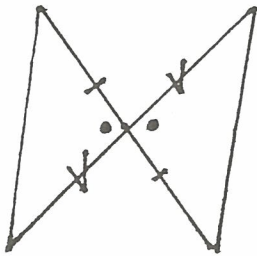
S.A.S

A.S.A.

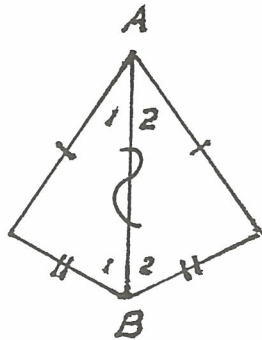
R.H.S

CONGRUENT

**SAME
SIZE** $\equiv$
OR $\cong$

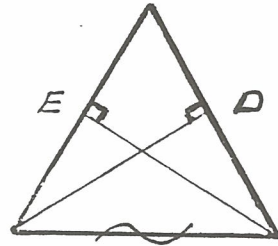


2 SIDES &
OPPOSITE ANGLE



PROVE $A_1 = A_2$
 $B_1 = B_2$

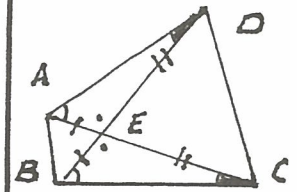
$AD = BE$



A B

PROVE $A = B$

D IS OPPOSITE 1
AND SO IS C



A IS OPPOSITE 11
AND SO IS B

PROPER ORDER
 $\triangle AED \equiv \triangle BEC$

S.A.S

S.S.S

R.H.S

$\triangle ABE \equiv \triangle BAD$

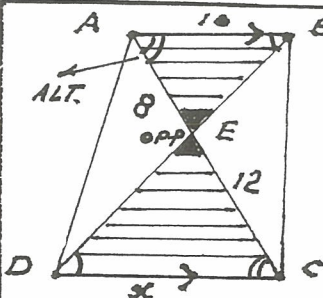
S.A.S

SAME ANGLES
DIFFERENT SIZE

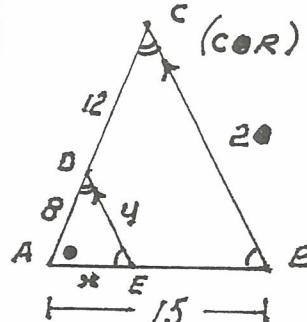
SIMILAR

**SAME
SHAPE** $\equiv$

ONLY SIDES OPPOSITE EQUAL ANGLES ARE IN PROPORTION

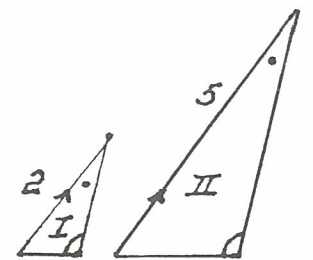
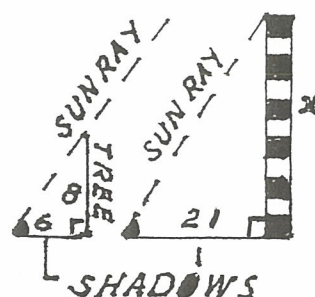


PROPER ORDER
 $\triangle ABE \equiv \triangle CDE$



$\triangle ADE \equiv \triangle ACB$

THE SHADOW SUM



RATIO

$$\begin{aligned} 8:12 &= 10:x \\ 8x &= 120 \\ x &= 15 \end{aligned}$$

$$\begin{aligned} y:20 &= 8:20 \\ y &= 8 \\ x:15 &= 8:20 \\ x &= 6 \end{aligned}$$

$$\begin{aligned} 8:x &= 6:21 \\ 6x &= 168 \\ x &= 28 \end{aligned}$$

$$A_I : A_{II} = 4:25$$

TRIANGLES

230.

AREA

2. WAYS DEPENDING ON THE DATA

WITH GEOMETRY

OR TRIGONOMETRY

$\frac{1}{2}$ BASE. HEIGHT

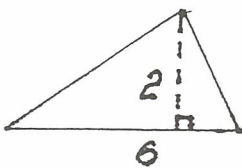
2 SIDES & INCLUDED ANGLE

$\frac{1}{2} ab \sin C$

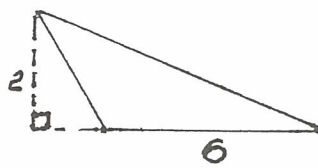
3 PAIRS POSSIBLE

INTERIOR
ALTITUDE

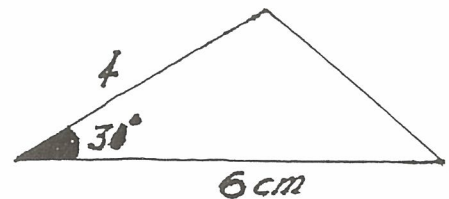
EXTERIOR
ALTITUDE



$A = 6$ SQ. UNITS



$A = 6$ SQ. UNITS



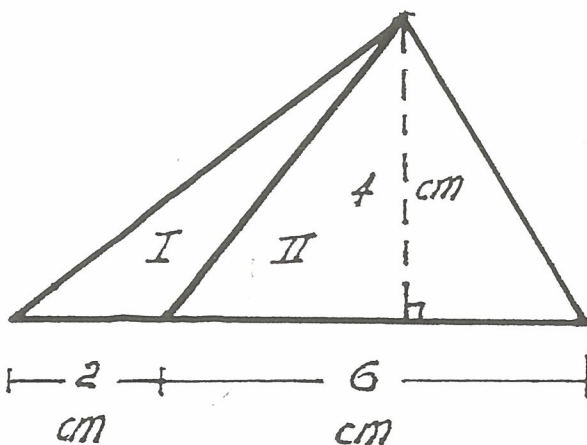
$A = 6 \text{ cm}^2$

RATIO AREAS

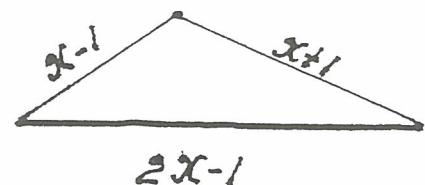
SAME ALTITUDE

$P = 23$
UNITS

SIDES
 $2x-1$
 $x+1$
 $x-1$



$A_I : A_{II} = 4 : 12 = 1 : 3$



MENTALLY

$$4x-1 = 23$$

$$x = 6$$

SIDES 11, 7 & 5 UNITS

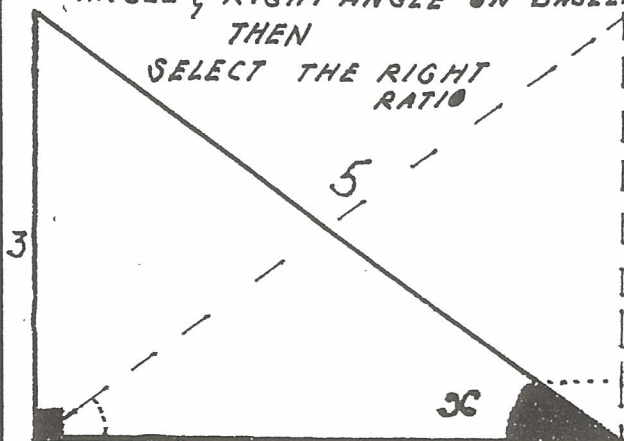
TRIGONOMETRIC RATIOS

231.

TRI GONOMETRY

TRI ANGLE MEASURE

TURN DIAGRAMS IF NECESSARY;
ANGLE & RIGHT ANGLE ON BASELINE.
THEN
SELECT THE RIGHT
RATIO

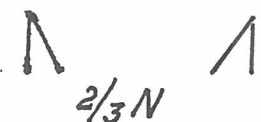


4

PROFESSIONAL MEMORY TRAINING:
VISUALISE TO REMEMBER
PERMANENTLY.
FORGET ABOUT THE SILLY SCHOOLWAY!

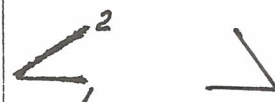
SINE

VISUALISE



LOSINE

FIRST THE FL
THEN THE RA



TANGENT



UPSIDE DOWN T

ANGLE TO RATIO

RATIO TO ANGLE

CASIO

CALCULATOR

CASIO

CALCULATOR

SIN 30

30 SIN
.5 (HALF)

← INVERSE
 $\sin x = \frac{3}{5} = .6$

INV. SIN
37°

COMPLEMENTS

COS 60

60 COS
.5

$\cos x = \frac{4}{5} = .8$

INV. COS
37°

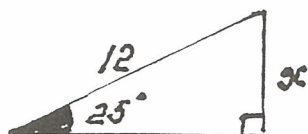
TAN 45

45 TAN
1

$\tan x = \frac{3}{4} = .75$

INV. TAN
37°

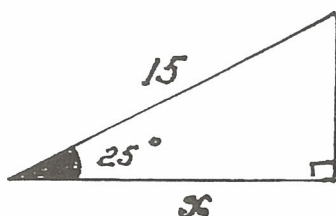
CALCULATING SIDES IN A RIGHT TRIANGLE NEEDS SELECTING THE RIGHT RATIO



$$\sin 25 = \frac{x}{12}$$

$$x = 12 \sin 25$$

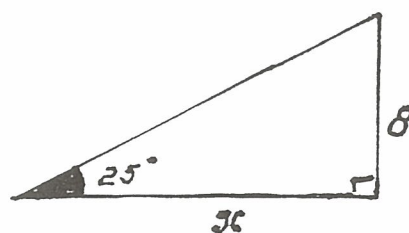
$$x \doteq 5$$



$$\cos 25 = \frac{x}{15}$$

$$x = 15 \cos 25$$

$$x \doteq 13.6$$

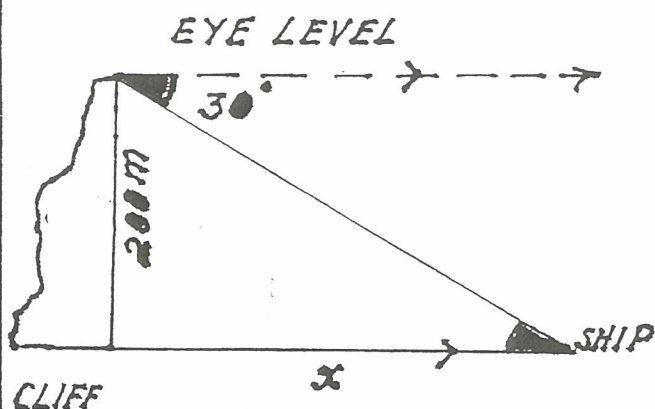


$$\tan 25 = \frac{8}{x}$$

$$x = 8 \div \tan 25$$

$$x \doteq 17$$

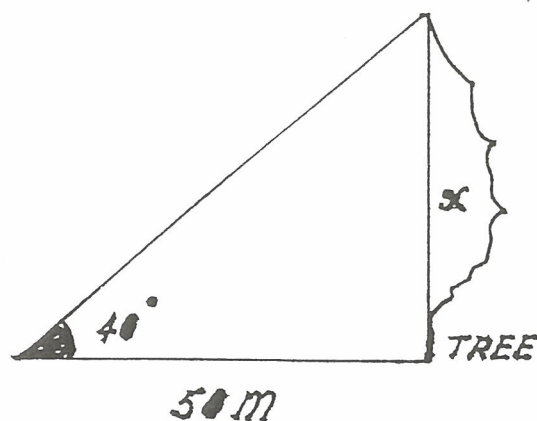
DEPRESSION



$$\tan 30 = \frac{200}{x}$$

$$x \doteq 346 \text{ m}$$

ELEVATION



$$\tan 40 = \frac{x}{50}$$

$$x \doteq 42 \text{ m}$$

TRIGONOMETRIC RATIOS

233

RECIPROCAL

ONE PREFIX C IN EACH PAIR. COMPARE PILOT-COPILOT

SINE

COSINE

TANGENT

BOTH RECIPR. & COMPLEMENT

COSECANT

SECANT

COTANGENT

USE

RECIPROCAL

KEY $1/x$

$$\sin 30 = .5$$

$$\text{COSEC } 30 = 2$$

$$\sec 60$$

$$60 \cos \text{INV } 1/x \ 2$$

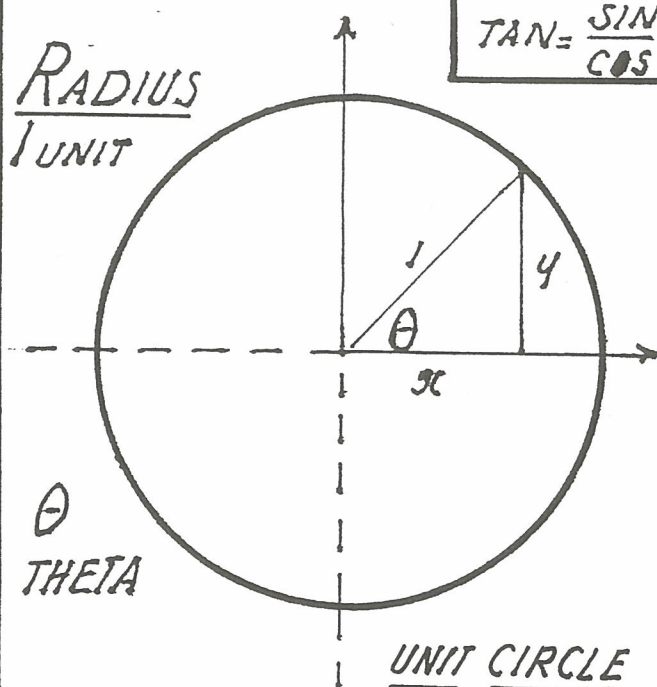
$$\cot 30$$

$$30 \tan \text{INV } 1/x \ 1.7$$

IDENTITIES

LEAVE OUT θ IN CALCULATIONS FOR SIMPLICITY!

RADIUS
1 UNIT



$$\tan = \frac{\sin}{\cos}$$

$$\sin \theta = y$$

$$\cos \theta = x$$

1. $\sin^2 + \cos^2 = 1$

2. $\text{DIVIDE BY } \cos^2 \theta$
 $\tan^2 + 1 = \sec^2$

3. $\text{DIVIDE NO. 1. BY } \sin^2$
 $1 + \cot^2 = \text{COSEC}^2$

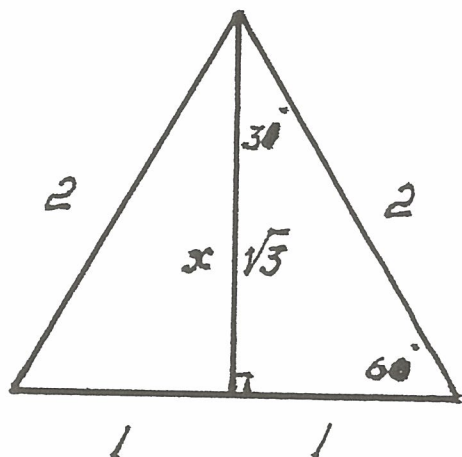
4. $\text{PROVE } \frac{2\cos^2}{1-\sin} = 2 + 2\sin$

$\frac{2(1-\sin^2)}{1-\sin}$	$\frac{2(1-\sin)(1+\sin)}{1-\sin}$	$2 + 2\sin$
	$\theta \text{ LEFT OUT!}$	

TRIGONOMETRIC RATIOS

234

THE EXACT RATIOS OF



PYTHAGORAS

$$x^2 = 3$$

$$x = \sqrt{3}$$

SIN

$$\frac{1}{2}$$

$$\frac{\sqrt{3}}{2}$$

COS

$$\frac{\sqrt{3}}{2}$$

$$\frac{1}{2}$$

TAN

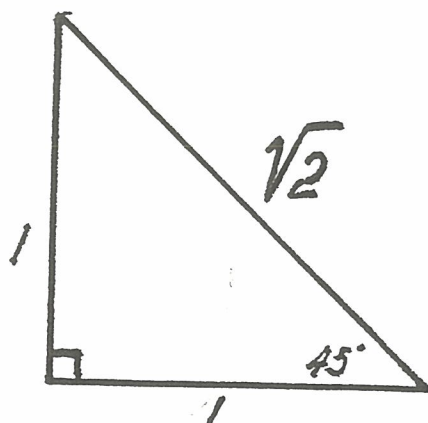
$$\frac{1}{\sqrt{3}}$$

$$\sqrt{3}$$

THE EXACT RATIOS OF

UNIT CIRCLE

CALCULATOR



SIN

$$\frac{1}{\sqrt{2}}$$

$$1$$

COS

$$\frac{1}{\sqrt{2}}$$

$$1$$

TAN

$$1$$

$$-\infty$$

$$\frac{\sin 90}{\cos 90} = \tan 90$$

$$= \frac{1}{0}$$

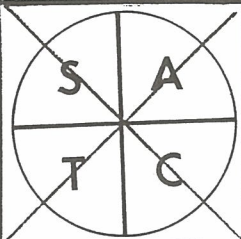


TRIGONOMETRIC RATIOS

235.

ALL PURPOSE SINE-COS CLOCK

INSTANT KNOWLEDGE AT YOUR EYE LIDS



WHY LEARN SILLY
MNEMONICS WHEN
THE UNDERSTANDING
IS IN YOUR EYE

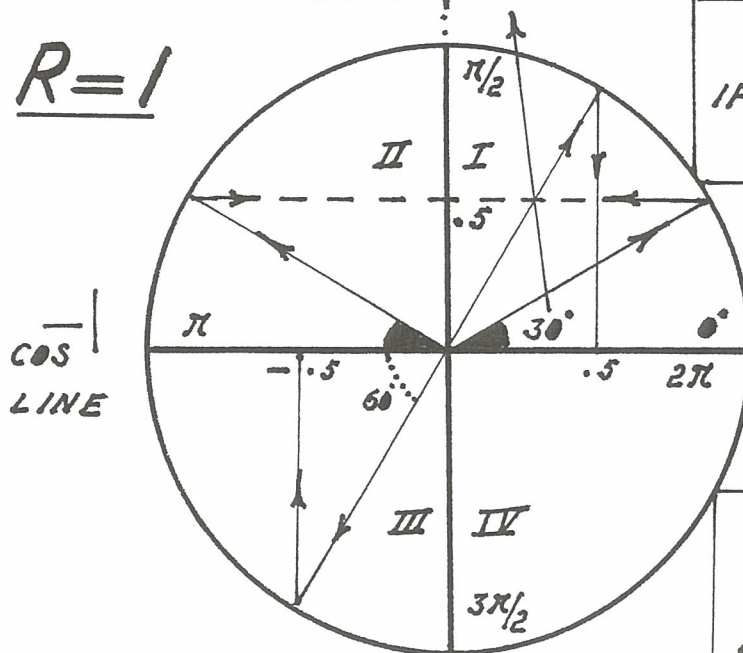
THE BRAIN CANNOT ABSORB PURE DATA

Any angle between 0° and 360° can be expressed
in one of the following forms where A is an acute
angle

1st quadrant	A	3rd quadrant	$180 + A$
2nd quadrant	$180 - A$	4th quadrant	$360 - A$

UNIT CLOCK : BASIC ANGLE

$$R=1$$



IF YOU WANT BREAD,
GO TO THE BAKER

SIMILARLY
IF YOU WANT SINE,
GO TO THE SINE LINE

SEE SINE
SEE COSINE
KNOW

$$TAN = \frac{SIN}{COS}$$

POS
NEG
 $-30^\circ = +330^\circ$

VISUALISE

MEANS
SEEING WITH
EYES CLOSED

SINE
LINE

LOOK & SEE

$$SIN 30^\circ = SIN 150^\circ = .5$$

$$COS 60^\circ = .5$$

$$COS 240^\circ = -.5$$

IN QUADRANT II,
SIN. IS POS } $\therefore$ TAN IS NEG
COS IS NEG }
BINGO!

TRIGONOMETRIC RATIOS

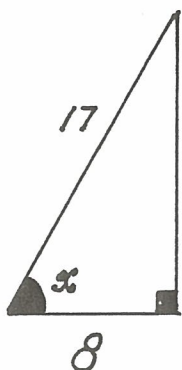
236.

$$\cos x = -\frac{8}{17}$$

$$\tan x > 0$$

$$x = 2 + 3 \cos \theta$$

$$y = 5 - 2 \sin \theta$$



RECOGNISE THE TRIAD

8, 15, 17

BASIC TRIANGLE
(ABSOLUTE VALUES)

FROM SINE-COS CLOCK

COS. NEG IN II & III

SINCE TAN IS POS, x IS IN III

LOOK FOR THE FIRST
2 STROKES OF N OR
MIRROR IMAGE

UPSIDE

DOWN T

$$\sin x = -\frac{15}{17}$$

$$\tan x = \frac{15}{8}$$

FROM PARAMETRIC TO
CARTESIAN (DESCARTES):
ELIMINATE THE PARAMETER θ

$$\cos \theta = \frac{(x-2)}{3}$$

$$\sin \theta = \frac{(y-5)}{-2}$$

USE THE
POPULAR
IDENTITY

$$\cos^2 + \sin^2 = 1$$

$$\frac{(x-2)^2}{9} + \frac{(y-5)^2}{4} = 1$$

$$P = \{x: \sqrt{3} \tan x = 2 \sin x\}$$

$$Q = \{x: 5 \tan^2 x + 13 = 11 \sec^2 x\}$$

WORKING WITHOUT x !

SIMPLICITY & EFFICIENCY

A PROGRAMMED TEACHER

ONCE SAID, "HEINOUS!" (EVIL)

$$\sec^2 = \frac{1}{\cos^2} = \frac{\sin^2 + \cos^2}{\cos^2} =$$

$$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \tan^2 + 1$$

$$\sqrt{3} \sin - 2 \sin \cos = 0$$

$$\sin(\sqrt{3} - 2 \cos) = 0$$

$$5 \tan^2 + 13 = 11(\tan^2 + 1)$$

$$6 \tan^2 = 2$$

$$\tan = \pm \frac{1}{\sqrt{3}}, \text{ BASIC } 30^\circ$$

VISUALISE SINE-COS CLOCK

$$\sin = 0$$

EXACT VALUES

$$\cos = \frac{1}{2} \sqrt{3}$$

$$Q = \{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$$

$$P \cap Q = \{30^\circ, 330^\circ\}$$

$$P = \{0^\circ, 30^\circ, 180^\circ, 330^\circ, 360^\circ\}$$

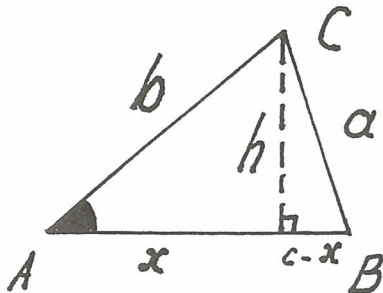
$$P \cup Q = \{0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ, 360^\circ\}$$

TRIGONOMETRIC RATIOS

237

COSINE RULE

FOR ACUTE AND OBTUSE TRIANGLES



PROOF FOR 1 SET

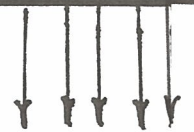
$$a^2 = h^2 + (c-x)^2$$

$$= h^2 + (c^2 - 2xc) + x^2$$

$$= b^2 + (c^2 - 2xc), \text{ SUB. } x = b \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

VISUALISE THE PATTERN, BUT



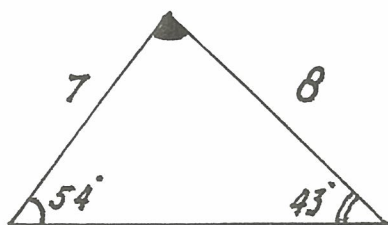
MENTALLY SUBSTITUTE AVAILABLE DATA DIRECTLY

OTHER WISE YOU HAVE TO FIGURE OUT WHICH IS WHICH

2 SIDES & BASE ANGLES

2 SIDES & INCLUDED ANGLE

INCLUDED ANGLE $C = 83^\circ$



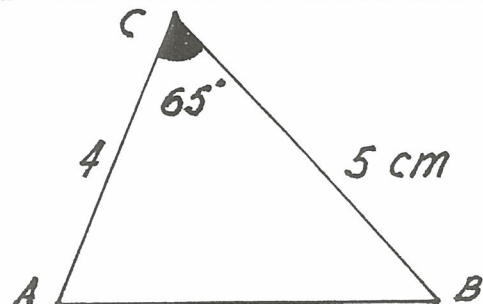
PUT DATA DIRECTLY IN THE CALCULATOR
THE BRAIN AND

IT IS EXTREMELY SILLY TO WRITE DOWN WHAT YOU'RE GOING TO DO! DO YOU WRITE THAT THE LIGHT IS RED OR DO YOU BRAKE?

EFFICIENCY (CASIO)

$$49 + 64 - 14 \times 8 \times 83 \cos = \dots$$

SQUARE ROOT, $x \div 10$



THE PROFESSIONAL WAY
PRESS:

$$16 + 25 - 40 \times 65 \cos = \dots$$

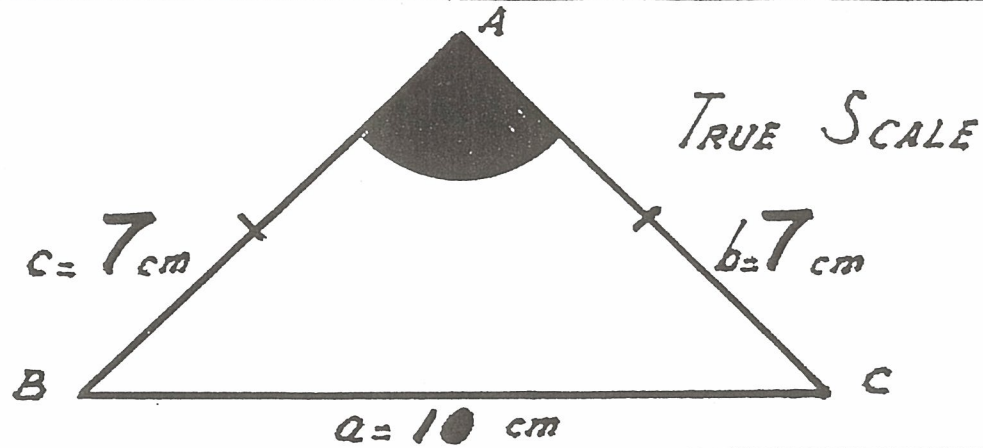
SQUARE ROOT, $AB \div 4.9 \text{ cm}$

2ND AREA FORMULA

FOR S.A.S: $\frac{1}{2} \times 4 \times 5 \sin 65$

THAT'S NOT THE SINE RULE

$$A = 10 \sin 65 \div 9 \text{ cm}^2$$



3 SIDES OR
2 SIDES & INCLUDED
ANGLE

OTHERWISE

SINE
RULE

COS RULE

SUBSTITUTION IS THEN DONE MENTALLY
AN IMPORTANT PART OF MEMORY TRAINING
IS TO VISUALISE !!
IF YOU CAN VISUALISE A \$100 NOTE,
YOU CAN VISUALISE A FORMULA.

IT ONLY
WORKS
IF YOU
WANT IT
TO WORK

$$a^2 = b^2 + c^2 - 2bc \cos A$$

CONVERSION

$$\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

EASY SQUARES & PRODUCTS ARE DONE MENTALLY

DIRECTLY IN CALCULATOR

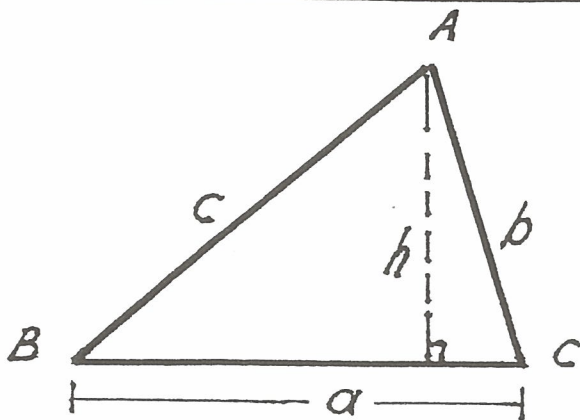
$$\cos A = (49 + 49 - 100) \div 98 = - \dots$$

UNKNOWN $\leftarrow$ PUT IN REVERSE $\rightarrow$ KNOWN
INV. COS $A \approx 91^\circ$

COS IN
QUADRANT
II & III
IS
NEGATIVE

SINE RULE

3 PAIRS



$$\sin B = \frac{h}{c}$$

$$h = c \sin B$$

$$\sin C = \frac{h}{b}$$

$$h = b \sin C$$

$$b \sin C = c \sin B$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

THIS SHOWS THE PATTERN TO USE

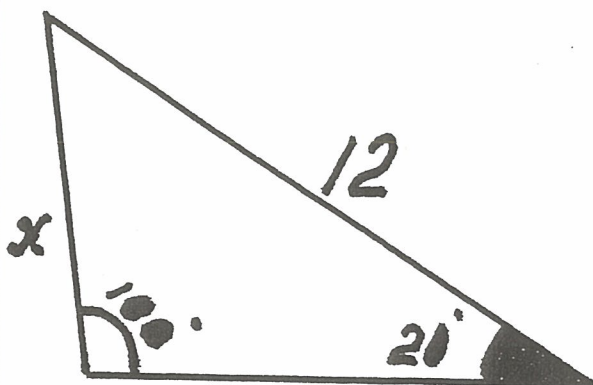
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

HOWEVER,

APPLICATION

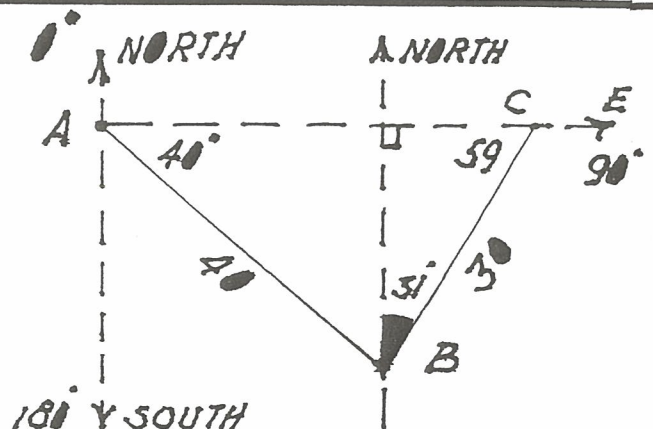
ONLY WORK WITH AVAILABLE DATA,
NOT WITH LETTERS THAT AREN'T THERE

PLANE FROM A 40 NAUTICAL
MILES, COURSE 130°. THEN 30
N.M. TILL EAST OF A (C)
FIND COURSE FROM B



$$\frac{x}{\sin 20} = \frac{12}{\sin 100} \quad \begin{matrix} \text{TIMES} \\ \text{DIVIDE} \end{matrix}$$

$$x = 12 \sin 20 \div \sin 100 \div 4$$



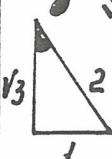
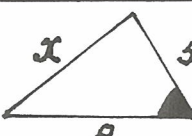
$$\frac{30}{\sin 40} = \frac{40}{\sin C}$$

$$40 \times \sin C \div 30 = \dots \text{INV SIN } 59$$

COURSE FROM B = 31°

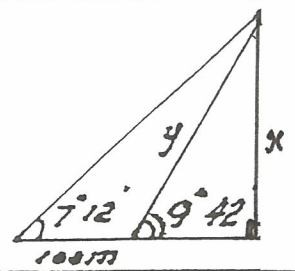
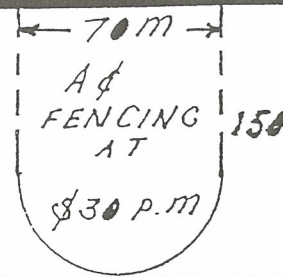
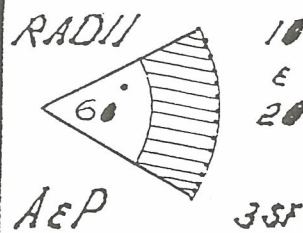
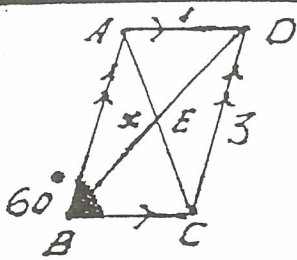
TRIGONOMETRY

240

$2\sin x + 1 = 0$	$\sin x + 2 = 0$	$\sin 30^\circ, 150^\circ, 210^\circ$	$\tan 160^\circ$
$0 \leq x < 2\pi$	$\sin x = -2$		
$\sin x = -\frac{1}{2}$ $x = 210^\circ \left(\frac{7\pi}{6}\right)$ $x = 330^\circ \left(\frac{11\pi}{6}\right)$	NO SOLUTION	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-.36$ Q II
$\operatorname{cosec} 250^\circ$	$\cot 65^\circ \sin 65^\circ$	$\sin^2 20^\circ + \sin^2 70^\circ$	$\cos x = -.7$
CASIO 250 SIN ... $\frac{1}{x}$ - 1.0642	$\cos 65^\circ$.42	$\sin^2 20^\circ + \cos^2 20^\circ$ /	$180 \leq x < 360^\circ$ QUADRANT III ONLY $x \doteq 225^\circ 34'$
$\sin x > .4$	$(\sin + \cos)^2 + (\sin - \cos)^2$	$\sin x = \frac{3}{4}$	$\cos x = \frac{1}{2}$
$0 \leq x < 360^\circ$ Q I & II $23^\circ 35' < x < 156^\circ 25'$	$2\sin^2 + 2\cos^2 =$ 2	$0^\circ < x < 90^\circ$  $\cos + \tan = \frac{19\sqrt{7}}{28}$	$0^\circ \leq x < 360^\circ$ $x = 60^\circ \text{ \& } 300^\circ$ $\sin x = \pm \frac{\sqrt{3}}{2}$ EXACT VALUES
$\cos x = \frac{4}{5}$	$2\cos x = \sqrt{3}$	$\tan x = \frac{1}{\sqrt{3}}$	S.A.S 5.60.8
$0 \leq x \leq \frac{\pi}{2}$ $\sin x = \frac{3}{5}$ $\cos 2x =$ $\cos^2 x - \sin^2 x =$ $\frac{7}{25} \quad (3U)$	$x = \frac{\pi}{6} = 30^\circ$ $x = \frac{11\pi}{6} = 330^\circ$ $0 \leq x < 2\pi$	$0^\circ \leq x < 360^\circ$  $x = 30^\circ$ $x = 210^\circ$	 $x = 7$ $A = 10\sqrt{3} \text{ u}^2$

TRIGONOMETRY

241.



$$AC = \sqrt{7}$$

$$BD = \sqrt{13}$$

$$A = \frac{1}{6} \times 300 \pi = 157.08$$

$$A = 12,424 \text{ m}^2$$

$$x = y \sin 9.42^\circ$$

$$\cos x = \frac{-8}{\sqrt{91}}$$

$$x = 147^\circ$$

$$P = \frac{1}{6} \times 60 \pi + 20 = 51.4 \text{ U}$$

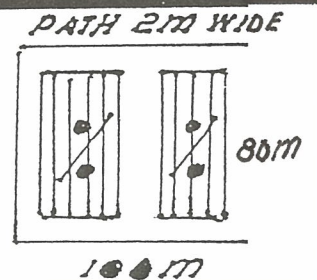
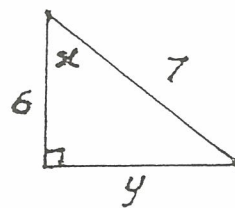
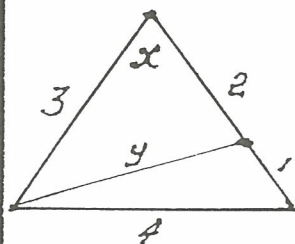
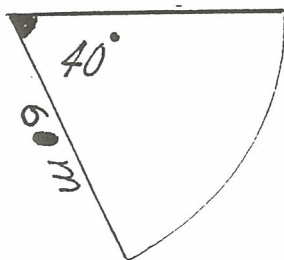
$$P = 480 \text{ m}$$

$$\text{COST } \$14,400$$

$$\frac{y}{\sin 71.2} = \frac{100}{\sin 23.6}$$

TIMES DIVIDE

$$x = 48.4 \text{ m}$$



$$A = \frac{1}{9} \times 3600 \pi = 1257 \text{ m}^2$$

$$\cos x = \frac{1}{9}$$

$$y^2 = 13 - \frac{12}{9}$$

$$y = 3.42$$

$$x = 31^\circ$$

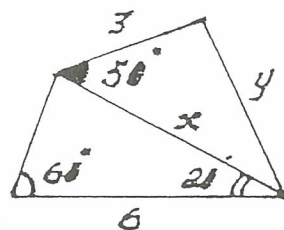
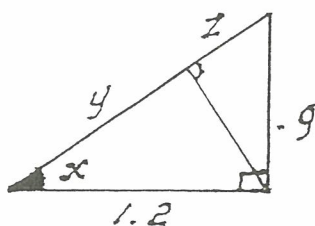
$$y = \sqrt{13} = 3.61$$

$$\text{TOTAL } A = 8000 \text{ m}^2$$

$$\text{PATH } 856$$

$$\text{GRASS } 7144$$

$$\div 80 = 89.3\%$$



$$a = 1.5$$

$$b = 2$$

$$c = 3$$

ΔABC
FIND $\hat{A}$

MOON

$$D = 3500 \text{ km}$$

OBSERVED AS 1°

$$x = 37^\circ$$

$$x = 5.28$$

$$y = 1.2 \cos x$$

$$= 1.2 \times \frac{12}{13} = .96$$

USE $x = 5.2763$
TO FIND y
(COS RULE)

$$y = 4.06$$

$$\cos A = \frac{10.75}{12}$$

$$A = 26^\circ$$

DISTANCE

$$1750 \div \tan \frac{1}{2}$$

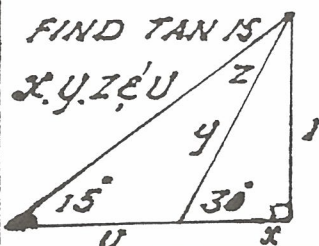
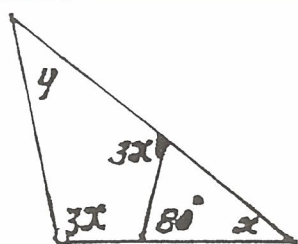
$$\text{APPR. } 200 \text{ } 530 \text{ km}$$

$$z = 1.5 - .96 = .54$$

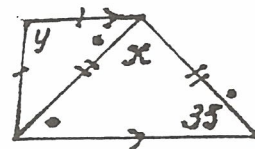
TRIAD 9 12 15

TRIGONOMETRY

242.



$a = 22$
 $b = 12$
 $c = 13$
 FIND A ($0 < A < 180$)



$80 + x = 3x$
 $x = 40^\circ$

$y = 2$
 $x = \sqrt{3}$
 $z = 15^\circ \therefore u = y$

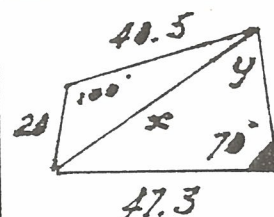
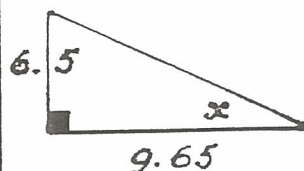
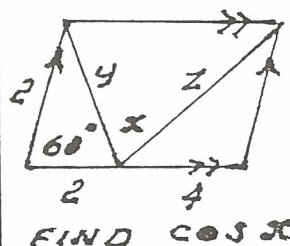
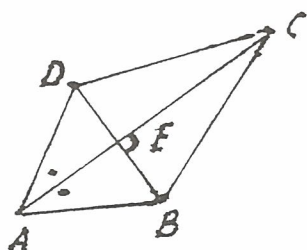
$\cos A = \frac{12^2 + 13^2 - 22^2}{(24 \times 13)}$
 $A = 123^\circ$

$x = 110^\circ$
 • ALTERNATE
 35°

$y = 260 - 6x = 20$

$\tan 15 = \frac{1}{2 + \sqrt{3}}$
 $= 2 - \sqrt{3}$

$y = 110^\circ$



$ABE \equiv ADE$ (ASA)
 $\therefore AD = AB$ &
 $DE = EB$
 $DEC \equiv BEC$ (SAS)
 $\therefore DC = BC$

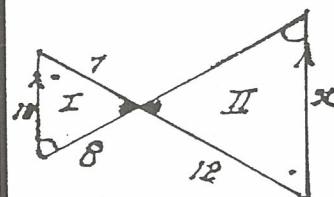
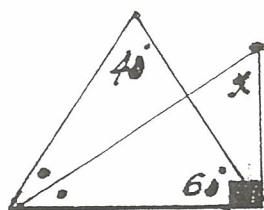
COS RULE
 $y = 2$
 $z = \sqrt{28} = 2\sqrt{7}$
 $\cos x = \frac{4 + 28 - 36}{8\sqrt{7}}$
 $= \frac{-\sqrt{7}}{14}$

$\tan x = 6.5 \div 9.65$

$x = 34^\circ$

COS RULE (SAS)
 $x = 48.2$

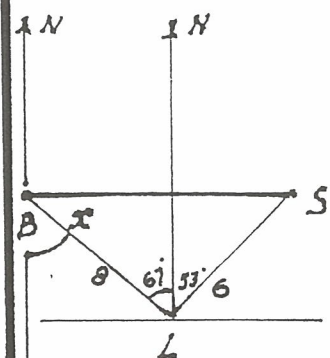
SINE RULE
 $y = 67^\circ$



$\angle S 153^\circ, 6 \text{ N.M}$
 $\angle B 293^\circ, 8 \text{ N.M}$
 SB & BEARING BS

$\cdot = 40^\circ$
 $x = 50^\circ$

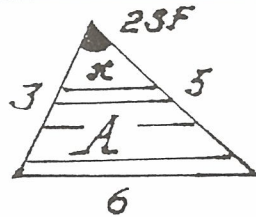
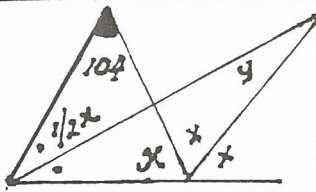
$I \parallel II$
 EQUAL ANGLES
 ALT. $\angle$ & OP.
 $\frac{10}{x} = \frac{7}{12}$
 $x = \frac{120}{7}$



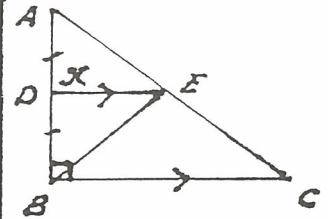
COS RULE
 $SB = 2\sqrt{37}$
 $\sin x = (3\sqrt{3}) \div (2\sqrt{37})$
 $x = 25^\circ$
 BEARING BS
 088°

TRIGONOMETRY

243.



$\frac{1}{6}$ ANNULUS
RADII
15 & 25 cm
A & P

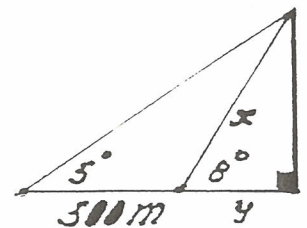
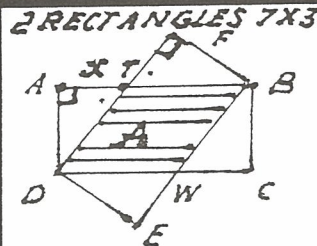
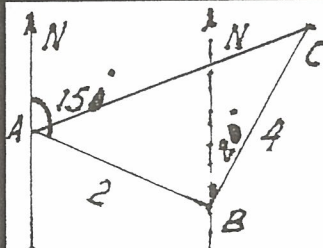
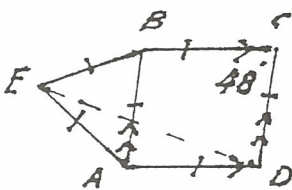


$$\begin{aligned} x &= 38^\circ \\ x &= 71 \\ + &= 19 \\ \hline &128 \\ y &= 52^\circ \end{aligned}$$

$$\begin{aligned} \cos x &= \frac{-2}{30} \\ x &= 94^\circ \text{ LEAVE} \\ \text{FULL FIGURE IN} \\ \text{THE CALCULATOR} \\ \sin \dots &\times 7.5 \\ A &= 7.5 \text{ U}^2 \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{6} \times 400 \pi \text{ cm}^2 \\ P &= \frac{1}{6} \times 80 \pi + 20 \\ &\text{cm} \end{aligned}$$

$$\begin{aligned} x &= 90^\circ (\text{COR}) \\ AED &\equiv BED (\text{SAS}) \\ \therefore BE &= AE \\ ADE \parallel ABC \\ \text{EQUAL ANGLES} \\ \therefore AE &= EC \\ \therefore BE &= EC \end{aligned}$$

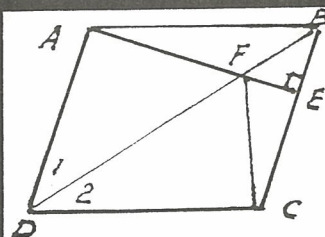
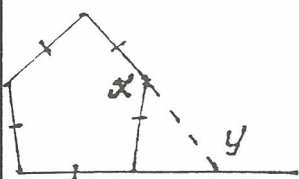


$$\begin{aligned} \hat{EAD} &= 108^\circ \\ \hat{EDA} &= 36^\circ \end{aligned}$$

$$\begin{aligned} AC^2 &= \\ 20 - 16 \cos 50 \\ AC &= 3.1 \end{aligned}$$

$$\begin{aligned} ADT &\equiv FBT (\text{AAS}) \\ TB &= 7 - x = TD \\ 9 + x^2 &= 49 - 14x + x^2 \\ x &= \frac{20}{7} \\ A &= 3(7 - \frac{20}{7}) \\ 12\frac{3}{7} \text{ U}^2 \end{aligned}$$

$$\begin{aligned} \frac{500}{\sin 3} &= \frac{x}{\sin 5} \\ x &= 833 \\ \text{TIMES DIVIDE} \\ \text{LEAVE } x &\text{ IN CALC.} \\ x \cos 8 &= y = \\ 820 (\text{NEAREST } 10) \end{aligned}$$



PROVE:
 $ADB \equiv CBD$
 $\hat{ADB} = \hat{CBD}$

PROVE:
 $AFD \equiv CFD$
 $\hat{FCD} = 90^\circ$

$$\begin{aligned} x &= \frac{1}{5} \times 540 = 108^\circ \\ y &= 2 \times 72 = 144^\circ \end{aligned}$$

RHOMBUS
ABCD

$$\begin{aligned} ADB &\equiv CBD (\text{SSS}) \\ D_1 &= D_2 \\ \hat{ADB} &= \hat{CBD} \\ (\text{ALTERNATE}) \end{aligned}$$

$$\begin{aligned} AFD &\equiv CFD (\text{SAS}) \\ \therefore \hat{DAF} &= \hat{DCF} \\ AE \perp BC &\therefore \\ AE \perp AD & \\ \therefore \hat{DCF} &= 90^\circ \end{aligned}$$

VARIATION

244.

96 km ON 16 L

540 km ON x L

COST C CENTS/km
VARIES AS THE CUBE
OF THE VELOCITY
 v km/hour

THE TWO QUANTITIES
ARE IN DIRECT
PROPORTION

$$C \propto v^3$$

$$C = K V^3$$

PETROL CONSUMPTION
VARIES DIRECTLY
WITH DISTANCE

DOUBLE VELOCITY.
COST MULTIPLIED BY 8

STEP 1. FIND THE CONSTANT K

$$\frac{96}{540} = \frac{16}{x}$$

$$x = 540 \times 16 \div 96$$

$$x = 90 \text{ L}$$

12 CENTS @ 60 km/h

$$12 = K \times 60^3 \therefore K = 1/18000$$

STEP 2. cost @ 90 km/h
is 40.5 CENTS/km

y VARIES INVERSELY
AS THE SQUARE OF x . FIND x
WHEN $y = 1/2$. $y = 2/9$ WHEN $x = 6$

LAW OF GRAVITATION
(NEWTON) JOINT VARIATION
MORE THAN 2 VARIABLES

STEP 1.

FIND K

$$y = \frac{K}{x^2}$$

$x = 6$
 $y = 2/9$ $K = 8$

THE FORCE OF ATTRACTION
BETWEEN 2 BODIES
VARIES DIRECTLY
AS THE PRODUCT
OF THE MASSES AND
INDIRECTLY AS
THE SQUARE OF THE DISTANCE

STEP 2.

SUBSTITUTE

$$\frac{1}{2} = \frac{8}{x^2}$$

$$x = \pm 4$$

$$F = \frac{K.m_1.m_2}{d^2}$$

DOUBLE m_1 ,
TREBLE m_2
HALVE d

F IS MULTIPLIED BY 24

VELOCITY & ACCELERATION KINEMATICS 245.

INTRODUCTION

"PARTICLE PROBLEMS"

t		TIME
s	}	DISPLACEMENT
x		DISTANCE FROM
$\dot{x}$	V	VELOCITY
$\ddot{x}$	a	ACCELERATION

V GIVES THE CHANGE OF x

a GIVES THE CHANGE OF V

DIFFERENTIATION $\longrightarrow$

$x \quad \dot{x} \quad \ddot{x}$

$\longleftarrow$ INTEGRATION

PARTICLE $s = 12 + 3t + 2t^3$

FUNCTION OF TIME

PARTICLE MOVES IN A STRAIGHT LINE (NOT ALONG A CUBIC CURVE)

$$S_4 = 152 \text{ m FROM ORIGIN}$$

$$V_4 = 3 + 6t^2 = 99 \text{ m/sec}$$

$$a_4 = 12t = 48 \text{ m/sec}^2 \text{ metres per second per second}$$

$$s = 4 - 3t + t^2 \quad s_0, s_2, V_2, s_{\text{MIN}} \text{ WHEN AT REST}$$

$$s = 8 + 12t - 4t^2 + \frac{1}{3}t^3 \quad s_0, V_0, V, a, \text{ REST}$$

$$s_0 = 4 \text{ m} \quad s_2 = 2 \text{ m} \quad V_2 = 1 \text{ m/s}$$

$$s_0 = 8 \text{ m} \quad V = 12 - 8t + t^2 \quad V_0 = 12 \text{ m/s}$$

AT REST WHEN $-3 + 2t = 0$
WHEN $t = 1.5 \text{ SEC.}$

$$a = -8 + 2t \quad a_2 = -4, a_6 = 4 \text{ m/s}^2$$

AT REST, $(t-6)(t-2) = 0$

$$s_{\text{MIN}}, \text{ WHEN } s' = V = 0, = 1.75 \text{ m}$$

$$\text{AT } t = 6 \text{ \& } t = 2$$

$$a = 2 + 5t \quad \text{INITIALLY AT REST AT } t=0 \quad \text{AT } t=0 \quad V=0$$

$$V = 20 \text{ m/s} \quad a = 8 - 2t \quad \text{STARTS AT 0 FIRST AT REST \& DISTANCE}$$

$$V_0 = 2t + \frac{5t^2}{2} + C = 0 \therefore C = 0$$

$$V_0 = 8t - t^2 + C = 20 \therefore C = 20$$

AT REST WHEN $(t-10)(t+2) = 0$, AT $t = 10 \text{ SEC}$

$$s_0 = t^2 + \frac{5t^3}{3} + C = 0 \therefore C = 0$$

$$s = 4t^2 - \frac{t^3}{3} + 20t, C = 0$$

$$s = t^2 + \frac{5t^3}{3}$$

$$s_{10} = 266\frac{2}{3} \text{ m FROM}$$

VELOCITY & ACCELERATION

246.

$$V_{\text{PARTICLE}} = 4t^3$$

$$x = 5 \text{ AT } t = 0$$

$$x \text{ AT } t = 2$$

$$x_t = t^4 + C$$

$$x_0 = 5$$

$$x_t = t^4 + 5$$

$$x_2 = 21$$

PARTICLE ALONG x-AXIS

$$x = 1 - \cos 2(t-1)$$

SHOW AT REST WHEN $t = 1$
& WHEN AND WHERE NEXT

$$x = 1 - \cos(2t-2)$$

FUNCTION OF A FUNCTION:
DER. PART X DER. WHOLE

$$V = 2 \sin(2t-2) = 0 \text{ (AT REST)}$$

$$2t-2 = 0, t = 1$$

NEXT WHEN $t = 1 + \frac{1}{2}\pi$
WHERE
 $x = 1 - \cos \pi = 2$

PARTICLE IN STRAIGHT LINE

$V = (12t - 3t^2) \text{ m/sec.}$ WHEN
AT REST? Q AT t_1 .
DISTANCE 4th. SECOND

PARTICLES: $V = 3 \sin 2t$

V_1 (3 S.F) & a_t

$a = (4 - 8/t^3) \text{ m.sec}^{-2}$.
 $V_2 = 10 \text{ m.sec}^{-1}$. FIND V_3

$$V = 12t - 3t^2 \text{ AT REST}$$

$$\text{WHEN } t(4-t) = 0,$$

$$\text{AT } t_0 \text{ \& } t_4$$

$$V_1 = 3 \sin .2 \doteq .596 \text{ m/sec}$$

$$(.2 \text{ RAD} = .2 \times \frac{180}{\pi})$$

$$a_t = 6 \cos 2t$$

$$\text{AT } t_1,$$

$$a = 12 - 6t = 6 \text{ m/sec}^2$$

$$a = 4 - 8t^{-3}$$

$$V = 4t + \frac{4}{t^2} + C$$

DISTANCE TRAVELLED IN 4th. sec

$$6t^2 - t^3 + C$$

$$(96 - 64) - (54 - 27) = 5m$$

$$x_4 - x_3$$

$$V_2 = 9 + C = 10 \therefore C = 1$$

$$V_3 = 13 \frac{4}{9} \text{ m/sec.}$$

VELOCITY & ACCELERATION

247.

PARTICLE $x = 3 + e^{2t}$

FIND: x_0, V_0, a_t

a IN TERMS OF x

$V_t = t^3 - 3t^2 + 4t - 6$ SHOW a_t MIN. = 1

$$x_0 = 4$$

$$V_t = 2e^{2t}, V_0 = 2$$

$$a_t = 4e^{2t}$$

$$e^{2t} = x - 3$$

$$a_x = 4(x - 3)$$

$$a_t = 3t^2 - 6t + 4, \dot{a} = 6t - 6, \ddot{a} = 6$$

MIN. $\dot{a} = 0 \therefore a_t = 1 \text{ m.sec}^{-2}$

PARTICLE $a_t = 12(t+3)^2$

$x_0 = 0$. FIND x_2

$$a_t = 12t^2 + 72t + 108$$

$$V_t = 4t^3 + 36t^2 + 108t + C$$

$$V_0 = 0 \therefore C = 0$$

$$x_t = t^4 + 12t^3 + 54t^2 + C$$

$$x = 0 \therefore C = 0$$

$$x_2 = 328$$

PARTICLE $x_t = t + 1 + \frac{1}{1+t}$

FIND: x_0, V_2, a_t

PARTICLE 2: $x_0 = 7, a_t = 2 + 6t$

FIND x_3

$$x_0 = 2$$

$$V_t = 1 - \frac{1}{(t+1)^2}, V_2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$a_t = \frac{2}{(t+1)^3}$$

$$a_t = 2 + 6t$$

$$V_0 = 0$$

$$x_t = t^2 + t^3 + C$$

$$V_t = 2t + 3t^2 + C$$

$$C = 0$$

$$x_0 = 7 \therefore C = 7$$

$$x_3 = 9 + 27 + 7 = 43$$

2 PARTICLES: $x_A = 12t + 5$
 $x_B = 6t^2 - t^3$. WHICH ONE FASTER
 AT $t = 1$. WHEN SAME SPEED. FIND
 a_B & MAX. POS x_B .

$$V_A = 12, V_B = 12t - 3t^2 = 9$$

SAME SPEED WHEN

$$3t^2 - 12t + 12 = 0$$

$$(t-2)^2 = 0, t = 2$$

$$a_B = 12 - 6t, a_B = -6$$

MAX. DISPLACEMENT B WHEN

$$12t - 3t^2 = 0, t(t-4) = 0, t = 4$$

$$x_{\text{MAX}} B = 32$$

VELOCITY & ACCELERATION

248.

PARTICLE: $x_t = 3 - \cos 2t$

$0 \leq t \leq 2\pi$. SKETCH.

P. AT REST: $t \in \pi$
FIND t FIRST $V_{\max}$.

$2t$	0	$\pi/2$	π	$3/2\pi$	2π	$5/2\pi$	3π	$7/2\pi$	4π
$\cos 2t$	1	0	-1	0	1	0	-1	0	1
x	2	3	4	3	2	3	4	3	2

t IN SEC. x IN m FROM $\rightarrow$
P. AT REST WHEN $2\sin 2t = 0$

$2t$	0	π	2π	3π	4π
t	0	$\pi/2$	π	$3/2\pi$	2π
x	2	4	2	4	2

MAX. SPEED WHEN $Q_t = 4\cos 2t = 0$
AT $t = \frac{1}{4}\pi$

PARTICLE: $x_t = t^3 - 9t$
FIND V_t, V_2 & TIME WHEN
P IS STATIONARY. $t \geq 0$

$$V_t = 3t^2 - 9$$

$$V_2 = 3$$

P STATIONARY
WHEN

$$t^2 = 3, t = \sqrt{3}$$

PARTICLE: $V_t = 4 - 2t$

$x_0 = 1$. FIND $x_t \in Q_t$ TIME
WHEN P AT REST. $|x|$ FIRST 4 SEC.

$$x_t = 4t - t^2 + 1$$

$$Q_t = -2 \text{ (CONSTANT)}$$

P. AT REST WHEN $t = 2$

$x = 5$, DISTANCE TRAVELLED
 $\rightarrow 4m$.

$x_4 = 1$, P HAS TRAVELLED
ANOTHER $4m \leftarrow$ TOTAL $8m$.

NUMBER OF BACTERIA

$$P_t = 1000 e^{0.007t} \text{ (t IN MIN)}$$

FIND: P_{10} &
RATE OF INCREASE

$$P_{10} = 1000 e^{0.07} \approx 1073$$

$$\frac{dP}{dt} = 7e^{0.007t} = 7e^{0.07} \approx 7.5$$

AFTER 10 MINUTES,
COLONY INCREASING
AT THE RATE OF
APPR. 7.5 BACTERIA PER MIN.

VELOCITY & ACCELERATION

249.

PARTICLE: $x_0 = 0, V_t = 2 - 3e^{-t}$
 FIND a_t & x_t , & P AT REST.
 SHOW x IS THEN $2/\ln \frac{3}{2} - 1$
 DISCUSS MOTION FOR t LARGE

$$x_t = 2t + \frac{3e^{-t} + C}{\text{BEWARE}} = 2t + 3e^{-t} - 3$$

$x_0 = 0, C = -3$

$$a_t = 3e^{-t} \text{ m/sec}^2$$

P AT REST WHEN
 $3e^{-t} = 2 \quad \& \quad t = \ln \frac{3}{2} !$

SUB x_t :

$$x_t = 2/\ln \frac{3}{2} - 1$$

t LARGE: $e^{-t} \rightarrow 0, x_t \rightarrow \infty$
 $V_t \rightarrow 2 \quad \& \quad a_t \rightarrow 0$

PARTICLE: $a_t = 4\pi^2 \cos \pi t$
 $x_0 = 0, V_0 = 2\pi \text{ m/sec}$. FIND x_t & V_t ,
 STATIONARY TIMES FOR $0 \leq t \leq 4$.
 SHOW THAT $x_{\text{MAX}} = 2(2 + \sqrt{3}) + 19\pi/3$

$$V_t = 4\pi \sin \pi t + C$$

$V_0 = 2\pi \therefore C = 2\pi$

$$V_t = 4\pi \sin \pi t + 2\pi$$

$$x_t = -4 \cos \pi t + 2\pi t + C$$

$x_0 = 0 \therefore C = 4$

$$x_t = -4 \cos \pi t + 2\pi t + 4$$

STATIONARY TIMES: $\sin \pi t = -\frac{1}{2}$
 $t = \frac{7}{6}, \frac{11}{6}, \frac{19}{6}, \frac{23}{6}$

MAX $x = 2(\sqrt{3} + 2) + 19\pi/3$
 $x_{19/6} = 2\sqrt{3} + 4 + 38\pi/6$

$x_{7/6} = 2\sqrt{3} + 4 + 14\pi/6$
 $x_{23/6} = -2\sqrt{3} + 4 + 46\pi/6$

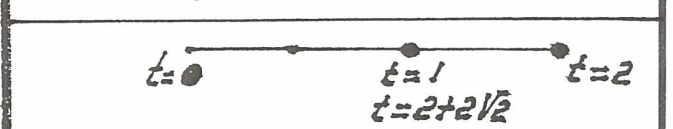
PARTICLE: $x_t = 3t^2/(4+t^3)^{-1}$
 $x_0 = 0$. FIND V_t , REST TIMES, x_{MAX} .
 SHOW x_1 & $x_{2+2\sqrt{2}}$ ARE EQUAL
 & DESCRIBE CHANGES OF x & MOTION $t \rightarrow \infty$

$$x_t = \frac{3t^2}{(4+t^3)} \quad \text{QUOTIENT RULE}$$

$$V_t = \frac{6t(4+t^3) - 9t^4}{(4+t^3)^2} = \frac{3t(8-t^3)}{(4+t^3)^2}$$

$V_0 = 3t(8-t^3) = 0, t = 0, t = 2$
 $x_{\text{MAX}} = 1$

$$x_{2+2\sqrt{2}} = \frac{3(2+2\sqrt{2})^2}{4+(2+2\sqrt{2})^3} = \frac{3}{5} = x_1$$



$t \rightarrow \infty, x \rightarrow 0$

PARTICLE: $x_t = t^3 - 21t^2$
 FIND a_t & STATIONARY TIMES

$$V_t = 3t^2 - 42t \quad \text{m/sec}$$

STATIONARY WHEN $3t(t-14) = 0$
 INITIALLY ($t=0$) & AT $t=14$ SEC.

$$a_t = 6t - 42 \quad \text{m/sec}^2$$

VELOCITY & ACCELERATION

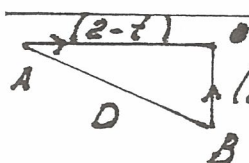
250.

2 CARS: V_A 1 km/min.

TRAVELLING EAST, NOW 2 km W. of θ

$V_B = V$ km/min, GOING N. NOW 1 km S. of θ
 V_B TO HIT A AT θ . MIN. DISTANCE AB

COLLISION AT θ , WHEN $V = \frac{1}{2}$



$$D^2 = (2-t)^2 + (1-Vt)^2$$

$$D^2 = 5 - 4t - 2Vt + t^2 + V^2t^2$$

$$\dot{D}^2 = -4 - 2V + 2t + 2V^2t = 0$$

$$\ddot{D}^2 = 2 + 2V^2 > 0 \therefore \text{MIN. WHEN}$$

$$t = \frac{2+V}{1+V^2}$$

2 PARTICLES: $a_P = 6 + e^{-t}$

$V_P = -1$, $x_P = 0$. FIND x_P

$x_Q = 2 \sin 5t + 3t^2 + 2$. PROVE $x_P < x_Q$
 EXPLAIN

$$V_t P = 6t - e^{-t} + C \quad \therefore C = 0$$

$$V_0 P = -1$$

$$x_t = 3t^2 + e^{-t} + C \quad \therefore C = -1$$

$$x_Q - x_P = 2 \sin 5t - e^{-t} + 3 = 2$$

$$\text{WHEN } t = 0, 1 < 2 \sin 5t + 3 < 5, \frac{1}{e^t} < 1$$

$$\therefore x_Q - x_P \text{ ALWAYS } > 0$$

SUB. IN D^2 , ETC.

UNSUITABLE FOR AN EXAM.

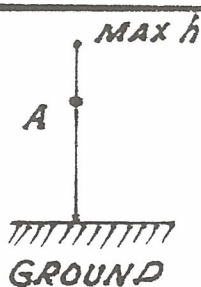
AT $t = 0$, Q IS 2 TO THE RIGHT OF P $\therefore$ IT WILL ALWAYS BE SO.

PARTICLE PROJECTED FROM A

$$h_t = \frac{49}{5}(3 + 2t - t^2). \text{ FIND: } h(A),$$

h_{MAX} , TIME TO HIT

THE GROUND



$$h_0 = h(A) = \frac{147}{5} \text{ m}$$

$$\text{MAX } h_t: \frac{49}{5}(2 - 2t) = 0$$

$$\text{WHEN } t = 1$$

SUB.

$$\text{MAX } h = \frac{196}{5} \text{ m}$$

WHEN PARTICLE HITS THE GROUND

$$(t^2 - 2t - 3) = 0$$

$$(t-3)(t+1) = 0, t = 3$$

PARTICLE: $x_0 = 0$,

$$V_t = 4(t - t^3) \text{ m/sec. FIND}$$

x_t , PARTICLE AT REST AGAIN

$$x_t = 2t^2 - t^4 + C$$

$$x_0 = 0 \therefore C = 0$$

$$x_t = 2t^2 - t^4$$

WHEN P IS AT REST AGAIN

$$4t(1 - t^2) = 0, t = 1$$

$$x_t = 1 \text{ m FROM START}$$

MATHEMATICS IN SCHOOLS

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"The chances for new ideas, different backgrounds and widening experiences are limited in a system that breeds its own teachers in its own schools and then gives them professional training in its own teachers' colleges."

"I found relatively little concern among teachers or inspectors to re-examine fundamentally their practices or assumptions.

I did not find a widespread eagerness to think hard and long about the theory of education."

From: ASSUMPTIONS
UNDERLYING
AUSTRALIAN
EDUCATION

R. FREEMAN BUTTS
Professor of Education
Teachers College,
Columbia
University, U.S.A.

Australian Council for Educational Research: 1955

NOTHING HAS CHANGED
ALTHOUGH CHANGE IS NO MORE DIFFICULT THAN
THE DECISION TO IMPLEMENT IT

For Whom The Alarmbells Toll

(Because nobody read the writing on the wall)

1. He who thinketh he leadeth and hath no one following him is only taking a walk.
2. Pareto Principle:
Teaching; 20 percent of the students take up 80 percent of the time.
Reading; 20 percent of the book contains 80 percent of the content.
3. By weakening the strong, we don't strengthen the weak.
4. We teach what we know. Unfortunately, we don't know what we don't know.
5. Efficiency is the foundation for survival.
6. Effectiveness is the foundation of success.
(Developing the leader within you: John C. Maxwell)
7. There is a paralysing Overchoice. (Future Shock: Alvin Toffler)
Many authors of Maths books have sold their soul to the Devil.

False propaganda and worn-out cliché's.

Emphasis is placed on the mastering of basic skills and their application so as to provide a sound basis for subsequent studies.

Books get thicker and thicker: In the Land of the Blind, the One-Eyed is King.

Exercises are presented in pairs so that students attempting odd or even numbers will cover a topic thoroughly. This also allows for extra questions to be used in remediation. Full revision exercises are provided.

Play now, Pay later. Blaa or Blaa Blaa or Blaa Blaa Blaa or Blaa Blaa Blaa Blaa.

Most topics are introduced assuming little or no pre-knowledge and we have tried, where possible, to stress the visual or concrete aspects of those topics.

Maths 7:486 pages.

We can write many sentences and patterns using different pronumerals.

Consider the following pattern:

one + one + one + one = 4 ones (or 4)

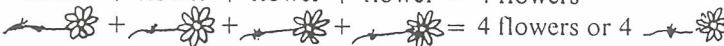
1 + 1 + 1 + 1 = 4 ones

two + two + two + two = 4 twos

3 + 3 + 3 + 3 = 4 threes

We could have the same pattern using words:

flower + flower + flower + flower = 4 flowers

or pictures  = 4 flowers or 4 

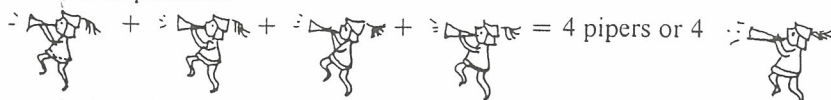
Using a letter as shorthand, we would have:

$f + f + f + f = 4 f$'s or $4f$

Similarly, the pattern of words:

piper + piper + piper + piper = 4 pipers

could be written as pictures:

 = 4 pipers or 4 

or mathematically using letters:

$p + p + p + p = 4 p$'s or $4p$

As we have seen in the above pattern, $a + a + a$ could be written $3 a$'s, but $3a$ is the very simplest way.

Similarly $x + x$ would be more simply written as $2x$.

Let the circle be unbroken?

Professional advice: Parents as partners in maths education. Don't leave it all to the teacher (he is too busy trying to understand the syllabus).

You have more time with your **child** than the teacher does. Play your part in assisting your **children** in their maths education. Maths **needs** to be seen as a high esteem subject.

Buy **your child** a calculator or a microcomputer with software that helps develop maths skills. They create more need to **understand** maths (not do).

Problem: Most adults admit with a certain pride that they **were** not good at Maths.

It is easy to understand why.

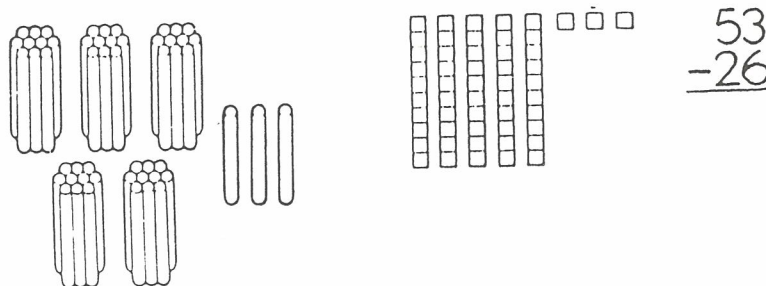
The masochist who introduced the following should see a psychiatrist.

1. The Decomposition Method

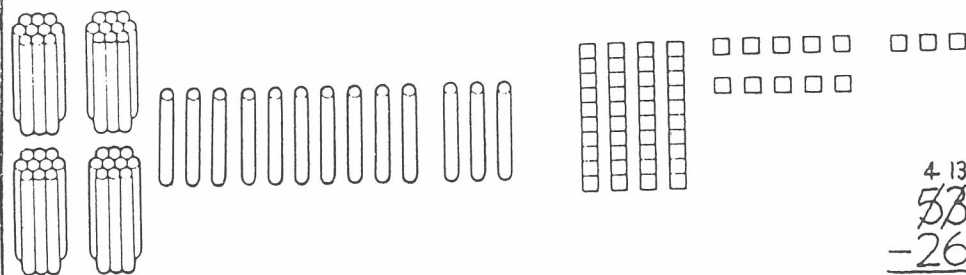
This method relies on a strong understanding of place value. Consider the example:

$$\begin{array}{r} 53 \\ -26 \\ \hline \end{array}$$

When using the decomposition method, we construct only the top line of the example using materials. Hence, using both bundling sticks and base 10 blocks, we have:



Ask the question 'Can we take 6 from 3?'. The answer to this question is no. To enable us to continue, we need to trade one of the tens from the 50 (5 tens) and break it into ones. This is set out in the following way:



With both the materials and in the written form, we now have 4 tens and 13 ones. We can take 6 ones from 13 ones to get 7 ones, and 2 tens from the remaining 4 tens to get 2 tens. In this way, the answer 27 is obtained.

8. It is not even a question of understanding any longer. We are now in the very dangerous next phase: **Who wants to understand?**

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Now consider
 PREMIS All zuks are floof.
 Zilch is floof.
 CONCLUSION Therefore Zilch is a zuk.
 Again, representing the first two statements as sets,

Thus, the conclusion is not necessarily true because Zilch can be a floof without being a zuk. If the premis (first statement) is false then it is not necessary to look at the syllogism's logic.

Some students still do as they're told. Most couldn't care less. Others set fire to schools. Many are on drugs and drink. A few commit suicide.

9. It is thought that **Attention Deficit Disorder** might be due to a mineral deficiency. What about TV?
 What about computers?
 What about this? **Just answers (no working).**

1. (a) 7 hats (b) 5 sheep (c) 5 dogs (d) 2 bananas (e) 4 sheep (f) 6 fish (g) 4 books (h) 8 cups
 (i) 3 lamps (j) 4 carrots (k) 5 crabs (l) 6 trees (m) 8 cats (n) 7 dogs (o) 3 rulers (p) 8 pigs
 2. (a) 3 (b) 3 (c) 2 (d) 6 (e) 5 (f) 7 (g) 4 (h) 4
 (i) 5 (j) 6 (k) 3 (l) 5 3. (a) 10a (b) 4l (c) 3x (d) 3x (e) 5y (f) 2m
 (g) 6m (h) 6d (i) 4c (j) 5y (k) 8d (l) 12l 4. (a) pig + pig + pig + pig + pig + pig
 (b) cat + cat + cat + cat + cat + cat + cat + cat + cat (c) rabbit + rabbit
 (d) mouse + mouse + mouse + mouse + mouse (e) dog + dog + dog + dog + dog
 (f) monkey + monkey + monkey (g) sheep + sheep + sheep (h) apple + apple + apple + apple
 (i) ruler + ruler + ruler + ruler (j) banana + banana
 5. (a) + + + + + (b) +

1. (a) 1000 whales (b) 1,000,000,000,000 tories (c) 10 dents (d) $\frac{1}{1000}$ of a cent (e) $\frac{1}{10}$ late
 (f) 1,000,000,000,000 wrists (g) 10 yings (h) $\frac{1}{1,000,000,000,000}$ of a ninny (i) 1,000,000 phones
 (j) $\frac{1}{1,000,000,000,000,000,000}$ of a knee (k) $\frac{1}{10}$ of a mate (l) 100 rings (m) 100 knells
 (n) $\frac{1}{1,000,000,000,000,000,000}$ of a miser (o) $\frac{1}{1000}$ of a yard (p) $\frac{1}{1,000,000,000,000}$ of a door. (q) $\frac{1}{1,000,000}$ of a cosmic
 (r) $\frac{1}{100}$ of a mental (s) $\frac{1}{1,000,000,000,000,000,000}$ of a ball (t) $\frac{1}{1,000,000,000,000}$ of a loss 2. google

10. **The cunning will inherit the Earth.**

The purpose of Maths in schools is creating, not expertise for students, but employment for teachers, tutors, tax agents and finance consultants.

11. **Ignorance is not bliss:** It runs the schools and ruins the country.

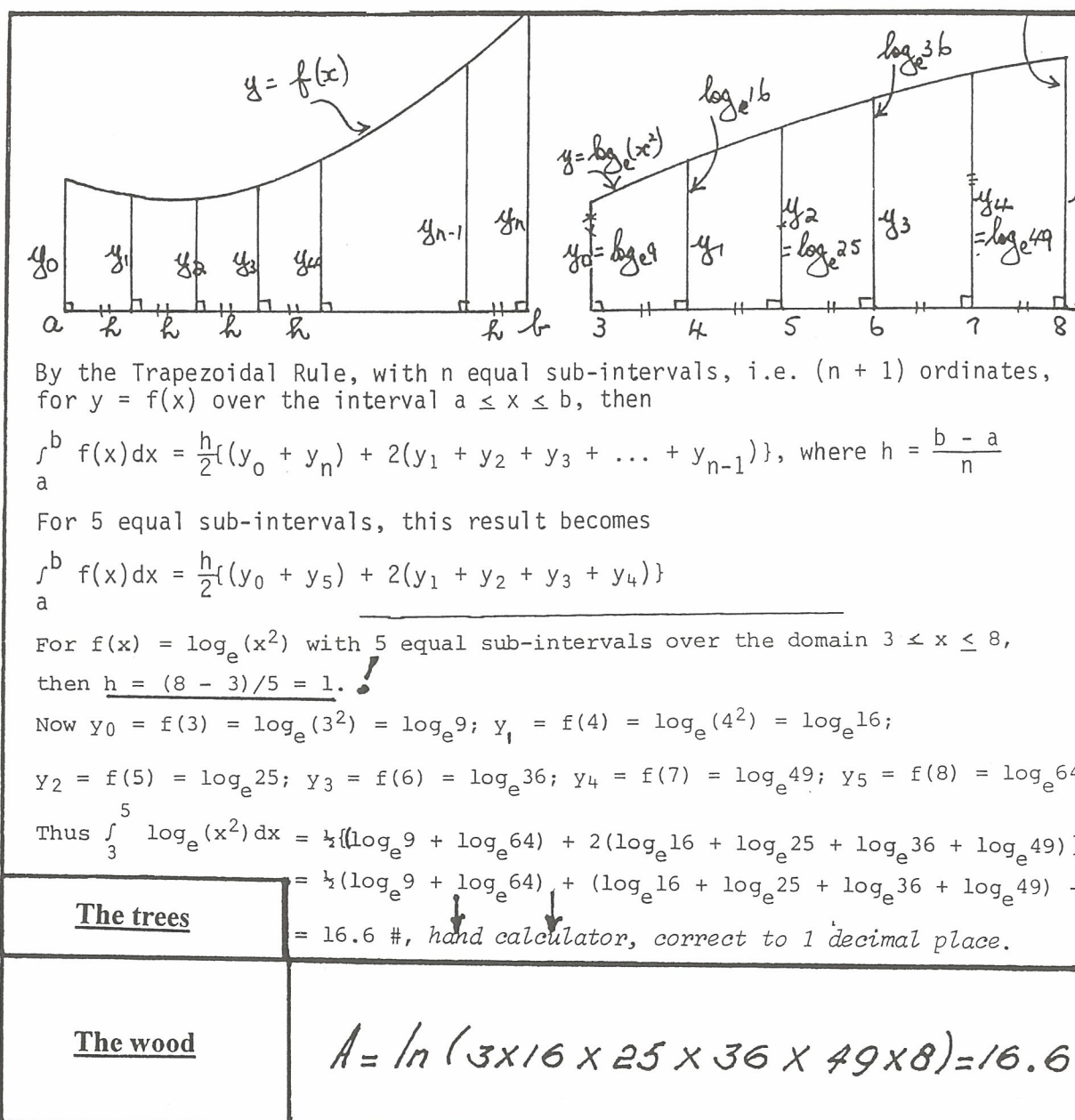
12. My academic-and professional background as well as my age allow me to speculate that I am one of the very few, semi-petrified relics of the past who really knows what "Back to Basics" means. My forthcoming, hocuspocus-free book for Primary Schools will prove it.
13. Most senior students are absolutely hopeless when it comes to basic arithmetic. A simple fraction, decimal or percentage is as difficult for them as more advanced Maths.
14. When little priorities demand too much of us, big problems arise. (Maxwell)
15. Too many priorities paralyse us. (Maxwell)

255.

A lion trainer holds a stool by the back and thrusts the legs towards the face of the wil animal. In an attempt to focus on all four, it becomes tame, weak and disabled because its attention is fragmented.

Hair-raising examples of compulsory incompetence
(All necessary working should be shown)

You can't see the wood for the trees



QUESTION 3

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- (i) (a) Find the equation of the straight line PQ, given that P has coordinates (3,4) and that PQ is parallel to the line whose equation is $y=2x+9$.
 (b) The line PQ intersects the x-axis at Q. Find the coordinates of the midpoint of PQ.

You may lead a horse to the water, but you cannot make it drink

3. (i) A diagram is not essential here but very useful to visualise the problem.

(a) Since the line $y = 2x+9$ is of form $y = mx+b$, then its slope $m = 2$ (and y-intercept $b = 9$).

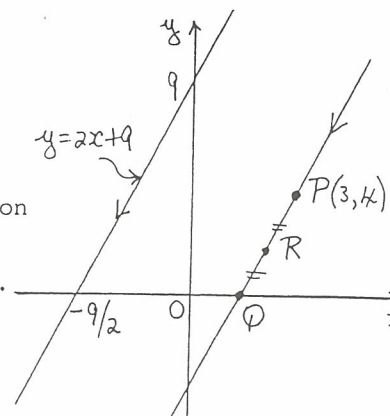
The reqd. line through P(3,4) is parallel to this and thus has the same slope $m = 2$. Using the line form $y - y_1 = m(x - x_1)$, then the reqd. line has equation

$y - 4 = 2(x - 3)$, i.e. $y - 4 = 2x - 6$, i.e. $y = 2x - 2$. (This may be written in general form $2x - y = 2$, if desired).

{Alternatively, all lines parallel to the line $y = 2x+9$ have equations of the form $y = 2x+k$, where k is a constant.

The required line passes through P(3,4) and thus (3,4) satisfies this equation i.e. $4 = 2(3) + k$, i.e. $4 = 6 + k$, i.e. $k = -2$

Hence the required line has equation $y = 2x - 2$ #



(b) This line cuts the x-axis ($y=0$) at Q where $0 = 2x - 2$, i.e. $x = 1$, and hence Q has coordinates (1,0).

Midpoint of PQ is given by $\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\}$ where $(x_1, y_1) \equiv (3, 4)$. Thus midpoint R (say) of PQ has coordinates $\{\frac{1}{2}(3+1), \frac{1}{2}(4+0)\} = \{\frac{1}{2}(4), \frac{1}{2}(4)\} = (2, 2)$ # and $(x_2, y_2) \equiv (1, 0)$

$$4 = 6 + b$$

$$y_{PQ} = 2x - 2$$

$$Q(1, 0)$$

$$M(2, 2)$$

BINGO!

(ii) Show that the second derivative of $\log_e(1 + \sin x)$ is $\frac{-1}{1 + \sin x}$.

Finding the needle in a haystack

(ii) We use the quotient rule $\frac{d}{dx}(\frac{u}{v}) = (v \frac{du}{dx} - u \frac{dv}{dx})/v^2$ and the identity $\cos^2 x + \sin^2 x = 1$ in the proof.

If $f(x) = \log_e(1 + \sin x)$

$$\therefore f'(x) = \frac{1}{1 + \sin x} \cdot \frac{d}{dx}(1 + \sin x) = \frac{1}{1 + \sin x} \cdot \cos x = \frac{\cos x}{1 + \sin x}$$

Thus $f''(x)$

$$= \frac{(1 + \sin x) \cdot \frac{d}{dx}(\cos x) - \cos x \cdot \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}, \text{ as a quotient}$$

$$= \frac{(1 + \sin x) \cdot -\sin x - \cos x \cdot \cos x}{(1 + \sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2}$$

$$= \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x} \#$$

$$y' = \frac{\cos x}{1 + \sin x}$$

$$y'' = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - 1}{(1 + \sin x)^2}$$

$$= \frac{-1}{1 + \sin x}$$

Q.E.D.

The haystack

The needle

(iii) Find the minimum value of $x \log_e x$ (where $x > 0$) correct to 3 decimal places.

From the sublime to the ridiculous is but a step

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(iii) Using the product rule, $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

If $y = x \log_e x$, then

$$y' = x \cdot \frac{d}{dx}(\log_e x) + \log_e x \cdot \frac{d}{dx}(x) = x \cdot \frac{1}{x} + \log_e x \cdot 1$$

$$= 1 + \log_e x$$

and $y'' = 0 + \frac{1}{x} = \frac{1}{x}$ ←

The ridiculous

Stationary point(s) on the curve occur where $y' = 0$

i.e. where $1 + \log_e x = 0$, i.e. $\log_e x = -1$, i.e. at $x = e^{-1}$

When $x = e^{-1}$, $y = e^{-1} \log_e e^{-1} = \frac{1}{e}(-1 \log_e e)$ using $\log a^n = n \log a$

$$= -\frac{1}{e}, \text{ since } \log_e e = 1$$

$$y'' = 1/e^{-1} = e > 0$$

Hence, the stationary point $(1/e, -1/e)$ is a rel. min. Now since $y'' = \frac{1}{x} > 0$ for all $x > 0$, $\therefore$ the curve is always concave up, and hence the minimum value of $x \log_e x$ is $-1/e$, i.e. $-e^{-1} \doteq -0.368$ #, correct to 3 decimal places.

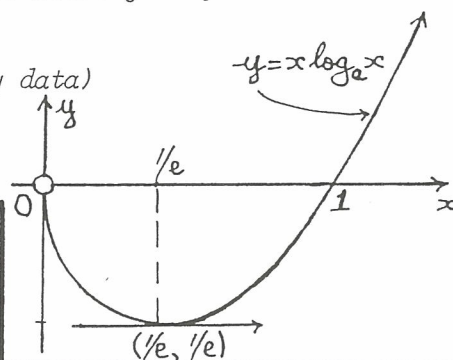
(Actually, the curve meets the x -axis where $y = 0$,

i.e. where $x \log_e x = 0$

i.e. where $\log_e x = 0$ (since $x \neq 0$ by data)

i.e. where $x = e^0 = 1$

A sketch of $y = x \log_e x$ is shown}



The sublime

$$\left. \begin{array}{l} y = x/\ln x \\ y' = 1/\ln x + 1 = 0 \end{array} \right\} \begin{array}{l} \text{MIN.} \\ y = -0.368 \end{array}$$

In $\triangle AKB$, by the Cosine Rule

$$\cos \theta = \frac{AK^2 + BK^2 - AB^2}{2 \cdot AK \cdot BK}, \text{ using form } \cos K = \frac{a^2 + b^2 - k^2}{2ab}$$

$$= \frac{(\frac{\sqrt{7}}{2})^2 + (\frac{\sqrt{13}}{2})^2 - (3)^2}{2 \cdot \frac{\sqrt{7}}{2} \cdot \frac{\sqrt{13}}{2}} = \frac{\frac{7}{4} + \frac{13}{4} - 9}{\sqrt{91}/2}$$

$$= \frac{5 - 9}{\sqrt{91}/2}, \text{ noting } \frac{7}{4} + \frac{13}{4} = \frac{20}{4} = 5$$

$$= \frac{-4}{1} \times \frac{2}{\sqrt{91}} = \frac{-8}{\sqrt{91}} \#$$

2 U H.S.C.!

The ridiculous

$$\int_0^4 y \, dx, \text{ where } y = 4 - \frac{1}{8} x^2$$

$$\int_0^4 (4 - \frac{1}{8} x^2) \, dx$$

$$[4x - \frac{1}{24} x^3]_0^4$$

$$[4 \cdot 4 - \frac{1}{24} \cdot 4^3] - 0$$

$$16 - \frac{8}{3}, \text{ noting } \frac{64}{24} = \frac{8}{3}$$

$$\frac{40}{3} \text{ units}^2, \text{ noting } 16 - \frac{8}{3} = \frac{48-8}{3}$$

2 U H.S.

The sublime

2. (i) If $2(x+3) = 3(x-1)$, $x=9$

2 U H.S.C.! then $2x+6 = 3x-3$

i.e. $2x-3x = -3-6$,

i.e. $-x = -9$, i.e. $x = 9$

10. (i) (a) $\int_0^1 (x^{1/5} - x^{1/3}) \, dx$

$$= \left[\frac{x^{6/5}}{6/5} - \frac{x^{4/3}}{4/3} \right]_0^1 = \left[\frac{5}{6} x^{6/5} - \frac{3}{4} x^{4/3} \right]_0^1$$

$$\rightarrow \left[\frac{5}{6} \cdot 1 - \frac{3}{4} \cdot 1 \right] - [0] = \frac{5}{6} - \frac{3}{4} = \frac{10-9}{12} = \frac{1}{12}$$

{Note $1^{6/5} = \sqrt[5]{1^6} = 1$ etc.} ← 2 U H.S.C.!

{+Note. The sum \$A to which \$P accumulates at r% p.a. compound interest over a period of n years is given by $A = P(1 + \frac{r}{100})^n$.

DIRECT

Thus, the sum to which \$3000 accumulates at 9% p.a. C.I. over a period of 28 years is $\$3000(1 + \frac{9}{100})^{28} = \$3000(1 + 0.09)^{28} = \$3000(1.09)^{28}$

$$3000 \times 1.09^{28}$$

7. (i) (a) Since $0 < x < \pi/2$, then x is an acute angle and we can form the triangle shown, where $\cos x = 4/5$.

TRIAD 3-4-5

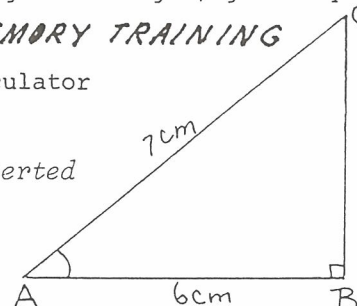
By Pythagoras' Theorem, $a = \sqrt{5^2 - 4^2} = \sqrt{9} = 3$.
Thus $\sin x = 3/5$ #

$$\sin x = \frac{3}{5}$$

(ii) (a) Using the trigonometric ratios for a right-angled triangle, given by the codeword SOH CAH TOA, then

$\cos A = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{6}{7}$, and using tables or a calculator then to the nearest degree, $A = 31^\circ$ #

{Note, if tables are used, then 6/7 must be first converted to a decimal 7)6.00000 correct to 4 places,
0.85714
i.e. $6/7 \div 0.8571$ }



$$6 \div 7 = \dots$$

INV.
COS
 31°

(b) In $\triangle APB$, using a form of the Cosine Rule adapted to the lettering of $\triangle APB$, namely $b^2 = p^2 + a^2 - 2pa \cos B$, then

$$\begin{aligned} AP^2 &= (4.907)^2 + (2)^2 - 2 \times 4.907 \times 2 \cos 60^\circ \\ &= 24.08 + 4 - 9.814 = 18.27 \\ \therefore AP &= \sqrt{18.27} \div 4.274 = 4.27 \text{ cm} \# \text{ (to 3 signif. figures)} \end{aligned}$$

$$\begin{aligned} &4.907^2 + 4 - 9.814 \\ &= \dots \sqrt{} \\ &4.274 \end{aligned}$$

(iii) Note $(1+3x)^2 > 0$ for $0 \leq x \leq 4/3$

Thus, the curve $y = 1/(1+3x)^2$ is above the x-axis for $x=0$ to $x=4/3$, and hence a sketch is not necessary
{+ See Note 1}

The required area A is given by

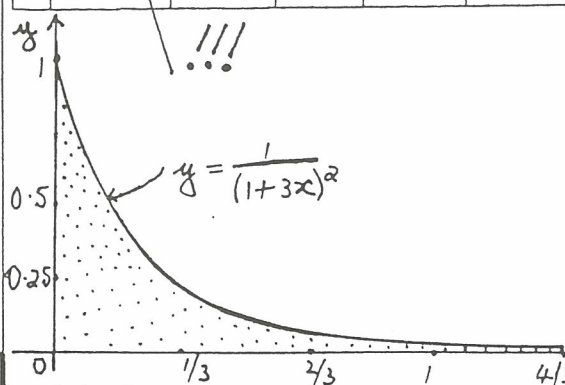
$$\begin{aligned} A &= \int_0^{4/3} y \, dx, \text{ where } y = 1/(1+3x)^2 \\ &= \int_0^{4/3} (1+3x)^{-2} dx, \text{ since } 1/a^2 = a^{-2} \\ &= \left[\frac{(1+3x)^{-1}}{-1.3} \right]_0^{4/3}, \text{ + See Note 2} \\ &= -\frac{1}{3} \left[\frac{1}{1+3x} \right]_0^{4/3}, \text{ since } a^{-1} = 1/a \\ &= -\frac{1}{3} \left\{ \frac{1}{1+4} - \frac{1}{1+0} \right\} \rightarrow -1 \times 3! \\ &= -\frac{1}{3} \left\{ \frac{1}{5} - 1 \right\} \\ &= -\frac{1}{3} \left(-\frac{4}{5} \right) \\ &= 4/15 \text{ units}^2 \# \end{aligned}$$

$$\left[\frac{1}{-3(1+3x)} \right]^{4/3}$$

$$\frac{-1}{15} + \frac{1}{3} = \frac{4}{15}$$

+ Note 1. If required, the curve $y = 1/(1+3x)^2$ could be sketched from a table of values thus:

x	0	1/3	2/3	1	4/3
	$\frac{1}{(1+0)^2}$ =1	$\frac{1}{(1+1)^2}$ =1/4	$\frac{1}{(1+2)^2}$ =1/9	$\frac{1}{(1+3)^2}$ =1/16	$\frac{1}{(1+4)^2}$ =1/25
y	1	0.25	0.11	0.06	0.04



+ Note 2

$$\begin{aligned} \frac{d}{dx} (1+3x)^{-1} &= -1 \cdot (1+3x)^{-2} \cdot \frac{d}{dx} (1+3x) \\ &= -1(1+3x)^{-2} \cdot 3 \end{aligned}$$

The Tyranny of Teachers

"We don't do it that way, it confuses the class."

"Write down the formula 10 times." (To my 3-Unit, year 12 student)

The Collapse of the "New Maths" Building

The obvious fact that reinforced concrete foundations last longer than those built with Q.S. (Quick sand) and B.S. had been ignored.

The hair-raising examples mentioned above illustrate clearly that "necessary working" should be interspersed with necessary retraining from kindergarten upwards despite the paperfall of merit certificates for the front door.

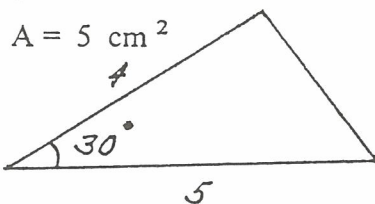
Doing senior Maths under the present conditions is like driving a car on four flat tyres, in first gear with handbrake still on.

Arithmetic is an abstract science.

Counting fingers, frogs and fish, cutting up oranges, ogres and ovals, playing hop-scotch and jump with decimal points or turning algorithms into a wrecking yard defeats the definition.

Formulas.

1. Professionals do not write down formulas; they apply them.

$3000 (1 + 0.08)^4$ which is $P (1 + r/100)^n$		<u>NOT:</u> $A = \frac{1}{2}ab\sin C$ $= \frac{1}{2} \times 4 \times 5 \times \sin 30$ $= 2 \times 5 \times \frac{1}{2}$ $= 1 \times 5$ $= 5 \text{ cm}^2$
--	---	--

To stop a student from developing his mental powers and his visualisation skills is a sin.

To stop a student from using his expertise in order to show the teacher that he didn't copy the answer, is even worse.

2. The medicine, not the script will cure a cold.
3. "The brain cannot absorb pure data; it becomes information only when it is seen through the spectacles of an idea." (de Bono)
That's why the traditional list of standard integrals is as useless to the well-prepared student as an Ethiopian dictionary to an Australian four-year old.
4. The existing, soon forgotten formulas for midpoint and interval should be scrapped and replaced by simple procedures based on existing knowledge. You don't need a formula to put on your shoes.

Warning.

T.V. - and computer programs artificially manipulate the brain into paying attention through the use of frequent visual and auditory changes. The cumulative effort produces the T.V. brain - a brain geared for television, but possibly to little else. (Endangered Minds: Jane M. Healy, Ph.D.)

Consequently, most students have lost the art of studying; the static black on white symbols don't register on them any more. Hence the projects and the assignments. They are not only time-consuming, but are also well because they involve thoughtless copying while lying on the floor having dinner in front of the

THE FUNCTION OF THE BRAIN

RUDOLF STEINER discussed it for different ages (imitation, memory, understanding).

EDWARD DE BONO formulated it.

NEUROSURGEONS discovered it, so it's now time to use that knowledge without procrastinating by waiting for still more information during pupil-free days.

T.V. downpour is like raindrops on a windscreen with wipers wiping warehouse wisdom. Without processing, brain cells die, and with them 400,000-year old Homo sapiens.

THE MISUNDERSTANDING OF UNDERSTANDING IN SCHOOLS

In Mathematics, understanding is of a scientific nature the necessity of which should not be exaggerated; of course it is.

In English, understanding is of a linguistic nature and should therefore be an inherent, though casual part of the subject; of course it is not.

That's why teachers of Mathematics might as well talk to a brick wall when they try to explain words like diameter, radius, equilateral, isosceles. Students' minds were never primed to pay attention to that sort of knowledge. Instead, they were made to absorb rows of synonyms, antonyms, male - female names of animals.

The following statement by a German sociologist is very significant and needs to be taken into account:

"Modern vocational training should emphasise comparatively abstract occupational and working qualities. The worker who controls automatic devices and instruments is required to display concentration, attention, high responsibility, technical knowledge, quick response, and reliability. These qualities must form in him a kind of permanent latent disposition - a kind of background on which to perform particular activities. Today, mainly abstract qualities are required of workers, such as the ability to organise, to handle people and to supervise, self-control, intelligence and reliability, exactitude, keeping up with work-pace, etc., while simple manual or intellectual knowledge and skills become less and less important.

The prevailing urge for specialisation should not deceive us. This urge has its origin not in the economy but in the people who look for jobs. Behind it, paradoxically enough, often lies the mistaken idea that the problem of occupational mobility can be solved by applying old means, i.e., by further specialising professional training. The concept of an occupation as a specialised activity evidently needs revising."

Besides being unique, the 2 Unit Maths Dictionary is the only book that does not present the subject as an end in itself; apart from being practical for some, it is primarily concerned with fostering the skills mentioned above.